The effect of numerical integration in nonmonotone nonlinear elliptic problems with application to numerical homogenization methods

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Abstract

A finite element method with numerical quadrature is considered for the solution of a class of second-order quasilinear elliptic problems of nonmonotone type. Optimal a priori error estimates for the H^1 and the L^2 norms are derived. The uniqueness of the finite element solution is established for a sufficiently fine mesh. Application to numerical homogenization methods is considered.

Résumé

Méthode d'éléments finis pour des problèmes elliptiques non-linéaires non-monotones : quadrature numérique et vitesses de convergence. On considère des méthodes d'éléments finis avec intégration numérique par quadrature pour des problèmes elliptiques quasi-linéaires de type non-monotone. Les vitesses de convergence optimales pour les normes H^1 and L^2 sont démontrées ainsi que l'unicité de la solution numérique pour un maillage suffisamment fin. Ces résultats permettent l'analyse multi-échelles de méthodes numériques d'homogénéisation.

Version française abrégée

L'objet de cet article est d'analyser l'influence des erreurs de quadrature pour la méthode des éléments finis appliquée à la classe d'équations elliptiques quasi-linéaire non-monotones (1). Ces équations interviennent dans de nombreux modèles, comme les problème d'infiltration d'eau dans des milieux poreux, des problèmes de potentiel électrique ou de diffusion de la chaleur dans des matériaux. Comme la méthode d'éléments finis fait intervenir des intégrales qui doivent en général être discrétisées par quadrature, il est important que l'analyse des vitesses de convergence de la méthode tienne compte des formules de quadrature utilisées.

Pour des problèmes elliptiques linéaires ou monotones, l'effet de l'intégration numérique est analysé dans [5,14] et [10]. Cependant, il n'existe à notre connaissance aucune analyse de vitesses de convergence pour des problèmes nonlinéaires de type nonmonotones. Dans [9], la convergence H^1 de la solution numérique est établie, mais sans vitesse de convergence et seulement pour des éléments finis linéaires par morceaux.

Sous des hypothèses usuelles sur le maillage pour des problèmes non-linéaires, sur la régularité des coefficients et des données, et sur les formules de quadrature (**Q1**),(**Q2**), également usuelles tant pour des problèmes avec intégration numérique (voir [5] ou [6, Sect. 29]) que pour des problèmes non-linéaires [9,15,8], nous prouvons des estimations optimales d'erreur pour les normes H^1 et L^2 de la méthode

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d'éléments finis (5), pour des élements simpliciaux ou quadrilatéraux d'ordre arbitraire. Nous prouvons également la convergence de la méthode de Newton dans un voisinage de la solution numérique lorsque le maillage est assez fin, ce qui est crucial pour une implémentation pratique. Ceci permet également de prouver l'unicité de la solution numérique.

Notre étude permet une analyse (avec discrétisation totale des échelles à la fois macroscopiques et microscopiques) d'une méthode d'homogénéisation numérique du type [1,8] pour une classe de problèmes non-linéaires d'homogénisation. Pour cette classe de problèmes, il n'existait jusqu' alors qu'une analyse semi-discrete et pour les dimensions $d \leq 2$. Notre analyse, avec discrétisation totale, permet de traiter la dimension $d \leq 3$. De plus, nous proposons une analyse de convergence (optimale) dans la norme L^2. Nous améliorons aussi l'estimation de l'erreur dite de résonnance et démontrons la convergence de la méthode de Newton utilisée pour calculer en pratique une solution du system non-linéaire. Plus de détails sur les résultats et l'analyse présentée ici sont donnés dans [2] (une seule échelle) et [3] (problèmes multi-échelles).

1. Introduction

We study finite element (FE) discretizations of second-order quasilinear elliptic problems of the form

$$-\nabla \cdot (a(x, u(x))\nabla u(x)) = f(x) \ \text{ in } \Omega, \qquad u(x) = 0 \ \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded convex polyhedron in $\mathbb{R}^d$ with $d \leq 3$. We make the following assumptions on the tensor $a(x, s) = (a_{mn}(x, s))_{1 \leq m, n \leq d}$:

– the coefficients $a_{mn}(x, s)$ are continuous functions on $\overline{\Omega} \times \mathbb{R}$ which are uniformly Lipschitz continuous with respect to s, i.e.,

$$\exists \Lambda_1 > 0, \quad |a_{mn}(x, s_1) - a_{mn}(x, s_2)| \leq \Lambda_1 |s_1 - s_2|, \ \forall x \in \overline{\Omega}, \forall s_1, s_2 \in \mathbb{R}, \forall \ 1 \leq m, n \leq d. \tag{2}$$

– $a(x, s)$ is uniformly coercive and bounded, i.e.,

$$\exists \lambda, \Lambda_0 > 0, \quad \lambda \|\xi\|^2 \leq a(x, s)\xi \cdot \xi, \qquad \|a(x, s)\xi\| \leq \Lambda_0 \|\xi\|, \qquad \forall \xi \in \mathbb{R}^d, \forall x \in \overline{\Omega}, \forall s \in \mathbb{R}. \tag{3}$$

Since (2)-(3) hold, it is known [12] that (1) has a unique solution $u \in H_0^1(\Omega)$ for all $f \in L^2(\Omega)$.

For linear or monotone elliptic problems, the effect of numerical quadrature in FEM has been analysed in [5,14] and [10]. To the best of our knowledge, there exist no analysis of the convergence rates for FEM with numerical quadrature applied to nonlinear problems of non-monotone type, as considered in this paper. In [9], the convergence in H^1 of the method is shown for piecewise finite elements, but without convergence rates. In the absence of numerical quadrature, optimal a priori error estimates in the H^1 and L^2 norms for FE methods (FEMs) were first given in [7].

Equations as (1) enter in the modeling of many important problems, we mention the infiltration of water in porous medium, the study of electrical potential or thermal diffusion in materials. As exact integration in FEMs is rarely possible, it is important to quantify the effect of numerical quadrature. Optimal convergence rates in the H^1 and L^2 norms are proved in this case. The practical implementation of the non-linear FEM requires a Newton method. We also establish the convergence of this latter method (crucial in applications) and the uniqueness of the FE solution for a sufficiently fine FE mesh. If $a(x, s)$ becomes independent of s, we recover the results of [5] on FEMs with numerical quadrature for linear problems (convex polyhedral domain case).

Application to numerical homogenization methods then considered. In contrast to previous results [8] obtained for nonmonotone homogenisation problems in dimension $d \leq 2$ (based on $2d$-Green function logarithmic estimates) for the H^1 norm and for a semi-discrete formulation, we obtain optimal convergence results for dimensions $d \leq 3$ and for a fully discrete method, which takes into account the microscale FE

discretization (see [1] il the context of linear problems). In addition, our results are also valid for arbitrary high-order elements of simplicial or quadrilateral type, optimal error estimates are obtained for the L^2 norm, and improved estimates are obtained for the resonance error [1] . More details on the results and the analysis presented here are given in [2] (one-scale problems) and [3] (multi-scale problems) .

2. Finite element method with numerical quadrature

We consider a family of partition $\mathcal{T}_h$ of Ω in simplicial or quadrilateral elements K of diameter h_K and denote $h := \max_{K \in \mathcal{T}_h} h_K$. We assume that this family of triangulation is conformal, shape regular and that it satisfies the inverse assumption

$$\frac{h}{h_K} \leq C \quad \text{for all } K \in \mathcal{T}_h \text{ and all } \mathcal{T}_h. \tag{4}$$

Notice that (4) is often assumed for the analysis of FEM for non-linear problems, see for instance [9,15,8]. We consider the family of FE spaces $V_h := \{v^h \in H_0^1(\Omega); \ v^h|_K \in \mathcal{R}^\ell(K), \ \forall K \in \mathcal{T}_h\}$, where $\mathcal{R}^\ell(K)$ is the space $\mathcal{P}^\ell(K)$ of polynomials on K of total degree at most ℓ if K is a simplicial FE, or the space $\mathcal{Q}^\ell(K)$ of polynomials on K of degree at most ℓ in each variable if K is a quadrilateral FE. We define a quadrature formula $\{\hat{x}_j, \hat{\omega}_j\}_{j=1}^J$ on a reference element $\hat{K}$, where $\hat{x}_j$ are integration points and $\hat{\omega}_j$ are quadrature weights. The quadrature formula $\{x_{K_j}, \omega_{K_j}\}_{j=1}^J$ is then defined as usual on any element K of the triangulation using a C^1-diffeomorphism. We make the following assumptions, which are similar to the case of linear elliptic problems (see [5] or [6, Sect. 29]):

(Q1) $\hat{\omega}_j > 0, \ j = 1, \dots, J, \quad \sum_{j=1}^J \hat{\omega}_j |\nabla \hat{p}(\hat{x}_j)|^2 \geq \hat{\lambda} \|\nabla \hat{p}\|_{L^2(\hat{K})}^2, \ \forall \hat{p}(\hat{x}) \in \mathcal{R}^\ell(\hat{K}), \quad \text{with } \hat{\lambda} > 0;$

(Q2) $\int_{\hat{K}} \hat{p}(x) dx = \sum_{j=1}^J \hat{\omega}_j \hat{p}(\hat{x}_j), \ \forall \hat{p}(\hat{x}) \in \mathcal{R}^\sigma(\hat{K})$, where $\sigma = \max(2\ell - 2, \ell)$ if $\hat{K}$ is a simplicial FE, or $\sigma = \max(2\ell - 1, \ell + 1)$ if $\hat{K}$ is a rectangular FE.

Consider for v, w scalar or vector functions that are piecewise continuous with respect to the partition $\mathcal{T}_h$ of Ω, the semi-definite inner product $(u, v)_h := \sum_{K \in \mathcal{T}_h} \sum_{j=1}^J \omega_{K_j} u(x_{K_j}) v(x_{K_j})$. The FE solution of (1) with numerical integration reads: find $u^h \in V_h$ such that

$$(a(\cdot, u^h)\nabla u^h, \nabla w^h)_h = F_h(w^h) \qquad \forall w^h \in V_h, \tag{5}$$

where the linear form $F_h(w^h)$ is an approximation of $\int_\Omega f(x) w^h(x) dx$ obtained for example by using a quadrature formula. If $f \in W^{\ell,q}(\Omega)$ with $1 \leq q \leq \infty$ and $\ell > d/q$, then f is continuous on $\overline{\Omega}$ and one can take $F_h(w^h) := (f, w^h)_h$. The existence of the FE solution $u^h \in V_h$ in (5) can be shown for all $h > 0$ using the Brouwer fixed point theorem. Details can be found for example in [7].

3. Convergence rates for FEM with numerical quadrature for nonlinear problems

Theorem 3.1 *[2] Consider u the solution of problem* (1). *Let $\ell \geq 1$. Let $d/\ell < q \leq \infty$. Let $\mu = 0$ or 1. Assume* **(Q1)**, **(Q2)**, (4), *and* [2]

$$u \in H^{\ell+1}(\Omega) \cap W^{1,\infty}(\Omega), \qquad a \in (W^{\ell+\mu,\infty}(\Omega \times \mathbb{R}))^{d \times d}, \qquad f \in W^{\ell+\mu,q}(\Omega).$$

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1. In Thm. 4.1, we obtain $r_{MOD} \leq C(\delta + \varepsilon/\delta)$ whereas $r_{MOD} \leq C(\delta + \sqrt{\varepsilon/\delta})$ has been obtain in [8, Thm. 5.5], where ε is the size of the period and δ is the size of the sampling domains.

2. Except for the $W^{1,\infty}$ assumption on u and the smoothness of $s \mapsto a(x,s)$ assumed to treat the non-linearity (as in [7]) the smoothness assumptions of Theorem 3.1 are identical to those classically assumed for linear problems [5],[6, Sect. 29].

In addition to (2), (3), assume that $\partial_u a_{mn} \in W^{1,\infty}(\Omega \times \mathbb{R})$, and that the coefficients $a_{mn}(x,s)$ are twice differentiable with respect to s, with the first and second order derivatives continuous and bounded on $\overline{\Omega} \times \mathbb{R}$, for all $m, n = 1 \ldots d$. [3]

Then there exists $h_0 > 0$ such that for all $h \leq h_0$, the solution u^h of (5) is unique, and the following H^1 and L^2 error estimates hold,

$$\text{if } \mu = 0, 1, \qquad \|u - u^h\|_{H^1(\Omega)} \leq Ch^\ell \qquad \text{for all } h \leq h_0, \tag{6}$$

$$\text{if } \mu = 1, \qquad \|u - u^h\|_{L^2(\Omega)} \leq Ch^{\ell+1} \qquad \text{for all } h \leq h_0, \tag{7}$$

where the constant C is independent of h.

Inspired by [7], the proof of Theorem 3.1 is conducted in three main steps.

Step 1. Using the compact injection $H^1(\Omega) \subset L^2(\Omega)$, the boundedness of a numerical solution in $H_0^1(\Omega)$ and the uniqueness in $H_0^1(\Omega)$ of the exact solution of (1), we show,

$$\|u - u^h\|_{L^2(\Omega)} \to 0 \text{ for } h \to 0. \tag{8}$$

Step 2. We derive the following H^1 a priori error bound

$$\|u - u^h\|_{H^1(\Omega)} \leq C(h^\ell + \|u - u^h\|_{L^2(\Omega)}), \text{ for all } h > 0. \tag{9}$$

The additional term $\|u - u^h\|_{L^2(\Omega)}$ in the right-hand side is due to the non-monotonicity of the differential operator of (1). The proof of (9) relies on an estimate for $(a(u^h)\nabla u^h, \nabla w^h) - (a(u^h)\nabla u^h, \nabla w^h)_h$ (obtained by using the Bramble-Hilbert lemma), uniform bounds for the semi-definite inner product $(v, w)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \sum_{j=1}^J \omega_{K_j} v(x_{K_j}) \cdot w(x_{K_j})$ (defined for piecewise continuous functions v, w) and the use of the Gagliardo-Niremberg inequality $\|v\|_{L^3(\Omega)}^2 \leq C\|v\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}$, which holds for all $v \in H^1(\Omega)$ for $d \leq 3$.

Step 3. We show that there exists $h_1 > 0$ such that

$$\|u - u^h\|_{L^2(\Omega)} \leq C(h^{\ell+\mu} + \|u - u^h\|_{H^1(\Omega)}^2), \text{ for all } h \leq h_1. \tag{10}$$

This estimates relies on an Aubin-Nitsche duality argument, where we consider the adjoint L^* of the linearized operator associated to (1), $L\varphi := -\nabla \cdot (a(\cdot, u)\nabla \varphi + \varphi \partial_u a(\cdot, u)\nabla u)$. An estimate between the FEM solution with numerical quadrature and the exact solution of this indefinite linear elliptic problem is needed, and obtained using the Schatz compacity argument [13].

Proof of the H^1 and L^2 estimates. Substituting (9) into (10) (with $\mu = 0$), we obtain

$$\|u - u^h\|_{H^1(\Omega)} \leq C(h^\ell + \|u - u^h\|_{H^1(\Omega)}^2), \text{ for all } h \leq h_1.$$

Substituting (8) into (9), we obtain $\|u - u^h\|_{H^1(\Omega)} \to 0$ for $h \to 0$. We deduce in the above inequality $1 - C\|u - u^h\|_{H^1(\Omega)} \geq \delta > 0$ for all $h \leq h_2$, with h_2 small enough (but independent of the particular solution u^h) hence, (6) is established for all $h \leq \min\{h_1, h_2\}$. The estimate (7) is deduced by substituting (6) into (10) with $\mu = 1$. The uniqueness of the FEM solution follows from Theorem 3.2. $\square$

Theorem 3.2 *Consider u^h a solution of (5). Under the assumptions of Theorem 3.1, there exist $h_0, \delta > 0$ such that if $h \leq h_0$ and $\sigma_h \|z_0^h - u^h\|_{H^1(\Omega)} \leq \delta$, then the sequence $\{z_k^h\}$ for the Newton method* [4]

$$N_h(z_k^h; z_{k+1}^h - z_k^h, v^h) = F_h(v^h) - (a(z_k^h)\nabla z_k^h, \nabla v^h)_h, \quad \forall v^h \in V_h, \tag{11}$$

is well defined, and

$$\|z_{k+1}^h - u^h\|_{H^1(\Omega)} \leq C\sigma_h \|z_k^h - u^h\|_{H^1(\Omega)}^2, \tag{12}$$

where C is a constant independent of h, k.

In the above theorem, $\sigma_h := \sup_{v^h \in V_h} \|v^h\|_{L^\infty(\Omega)}/\|v^h\|_{H^1(\Omega)}$. Using (4), on can show the standard estimates $\sigma_h \leq C(1 + |\ln h|)^{1/2}$ for $d = 2$, and $\sigma_h \leq Ch^{-1/2}$ for $d = 3$, where C is independent of h.

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3. The uniqueness of u^h for all $h \leq h_0$ and the H^1 estimate (6) for all $h > 0$ both hold without this additional assumption and without (4) and $u \in W^{1,\infty}(\Omega)$, if $C\lambda^{-1}\Lambda_1 \|u\|_{H^2(\Omega)} < 1$ (where C depends only on Ω and $(V_h)_{h>0}$).

4. We define $N_h(z^h; v^h, w^h) := (a(\cdot, z^h)\nabla v^h, \nabla w^h)_h + (v^h \partial_u a(\cdot, z^h)\nabla z^h, \nabla w^h)_h$..

4. Application to numerical homogenization.

We consider a class of nonlinear nonmonotone multiscale problems of the form (1) with a $d \times d$ tensor $a^\varepsilon(x, u)$ satisfying (2), (3) uniformly in ε. Here ε represent a small scale in the problem. The following homogenization result is shown in [4, Theorem 3.6]: there exists a subsequence of $\{a^\epsilon(\cdot, s)\}$ (again indexed by ε) such that the corresponding sequence of solutions $\{u_\varepsilon\}$ converges weakly to u_0 in $H^1(\Omega)$, where u_0 is solution of a so-called homogenization problem

$$-\nabla \cdot \left(a^0(x, u_0(x))\nabla u_0(x)\right) = f(x) \text{ in } \Omega, \qquad u_0(x) \text{ on } \partial\Omega, \tag{13}$$

with a homogenized tensor $a^0(x, s)$ which can be shown to have similar properties as assumed for $a^\varepsilon(x, s)$.

The FE-HMM method for computing a numerical approximation u^H of u_0, essentially similar to the method proposed in [8] [5] reads as follows. It is based on a macroscopic FEM defined on QF with a macro FE space $S_0^\ell(\Omega, \mathcal{T}_H)$ defined as the FE space described in Sect. 2, and microscopic FEMs recovering the missing macroscopic tensor at the macroscopic quadrature points. For each macro element $K \in \mathcal{T}_H$ and each integration point $x_{K_j} \in K$, $j = 1, \ldots, J$, we define the sampling domains $K_{\delta_j} = x_{K_j} + (-\delta, \delta)^d$, $(\delta \geq \varepsilon)$. For each K_{δ_j}, we then define a micro FE space $S^q(K_{\delta_j}, \mathcal{T}_h) \subset W(K_{\delta_j})$ with simplicial or quadrilateral FEs and a conformal and shape regular family of triangulation $\mathcal{T}_h$. The space $W(K_{\delta_j})$ is either the Sobolev space $W(K_{\delta_j}) = W_{per}^1(K_{\delta_j}) = \{z \in H_{per}^1(K_{\delta_j}); \int_{K_{\delta_j}} z \, dx = 0\}$ for a periodic coupling or $W(K_{\delta_j}) = H_0^1(K_{\delta_j})$ for a coupling through Dirichlet boundary conditions.

FE-HMM Find $u^H \in S_0^\ell(\Omega, \mathcal{T}_H)$ such that $B_H(u^H; u^H, w^H) = F_H(w^H), \forall w^H \in S_0^\ell(\Omega, \mathcal{T}_H)$, where

$$B_H(u^H; v^H, w^H) := \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K_j}}{|K_{\delta_j}|} \int_{K_{\delta_j}} a^\varepsilon(x, u^H(x_{K_j})) \nabla v_{K_j}^{h, u^H(x_{K_j})}(x) \cdot \nabla w_{K_j}^{h, u^H(x_{K_j})}(x) dx, \tag{14}$$

and $w_{K_j}^{h, u^H(x_{K_j})}$ (and similarly for $v_{K_j}^{h, u^H(x_{K_j})}$) denotes the solution of the following micro problem (15) with parameter $s = u^H(x_{K_j})$. Find $w_{K_j}^{h,s}$ such that $w_{K_j}^{h,s} - (w^H(x_{K_j}) + (x - x_{K_j}) \cdot \nabla w^H(x_{K_j})) \in S^q(K_{\delta_j}, \mathcal{T}_h)$ and

$$\int_{K_{\delta_j}} a^\varepsilon(x, s) \nabla w_{K_j}^{h,s}(x) \cdot \nabla z^h(x) dx = 0 \quad \forall z^h \in S^q(K_{\delta_j}, \mathcal{T}_h). \tag{15}$$

We make the following smoothness and structure assumptions on the tensor.
(**H1**) Given $q \in \mathbb{N}$, the cell functions $\psi_{K_j}^{i,s} \in W(K_{\delta_j})$ such that

$$\int_{K_{\delta_j}} a^\varepsilon(x, s) \nabla \psi_{K_j}^{i,s}(x) \cdot \nabla z(x) dx = -\int_{K_{\delta_j}} a^\varepsilon(x, s) \mathbf{e}_i \cdot \nabla z(x) dx, \ \forall z \in W(K_{\delta_j}). \tag{16}$$

satisfy the bound $|\psi_{K_j}^{i,s}|_{H^{q+1}(K_{\delta_j})} \leq C \varepsilon^{-q} \sqrt{|K_{\delta_j}|}$, with C independent of ε, the quadrature point x_{K_j}, the domain K_{δ_j}, and the parameter s for all $i = 1 \ldots d$. Here, $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denotes the canonical basis of $\mathbb{R}^d$. The same assumption also holds with the tensor a^ε replaced by $(a^\varepsilon)^T$ in (16).
(**H2**) for all $m, n = 1, \ldots, d$, we assume $a_{mn}^\varepsilon(x, s) = a_{mn}(x, x/\varepsilon, s)$, where $a_{mn}(x, y, s)$ is y-periodic in Y, and the map $(x, s) \mapsto a_{mn}(x, \cdot, s)$ is Lipschitz continuous and bounded from $\overline{\Omega} \times \mathbb{R}$ into $W_{per}^{1,\infty}(Y)$.

Following the framework presented in Sect. 3, we obtain the following H^1 and L^2 a priori estimates.

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5. In [8] (14) is based on exact micro functions v_{K_j}, w_{K_j} instead of the FE micro functions $v_{K_j}^{h,s}, w_{K_j}^{h,s}$ and the micro-problems are nonlinear (see [8, equs. (5.3)-(5.4)]).

Theorem 4.1 *[3] Let $\ell \geq 1$, $q \geq 1$ and $\mu = 0$ or 1. In addition to the assumptions of Theorem 3.1 on problem (13), assume (**H1**), (**H2**), and assume that a^ε satisfies (2), (3). Then, there exist $H_0 > 0$ and $r_0 > 0$ such at if $H \leq H_0$ and $h/\varepsilon \leq r_0$ then*

$$\|u_0 - u^H\|_{H^{1-\mu}(\Omega)} \leq \begin{cases} C(H^{\ell+\mu} + (\dfrac{h}{\varepsilon})^{2q} + \delta), & \text{if } W(K_{\delta_j}) = W^1_{per}(K_{\delta_j}) \text{ and } \dfrac{\delta}{\varepsilon} \in \mathbb{N}, \\[3mm] C(H^{\ell+\mu} + (\dfrac{h}{\varepsilon})^{2q}), & \text{if } W(K_{\delta_j}) = W^1_{per}(K_{\delta_j}) \text{ and } \frac{\delta}{\varepsilon} \in \mathbb{N}, \text{ and } a^\varepsilon(x, s) \\ & \text{is replaced by } a(x_{K_j}, x/\varepsilon, s) \text{ in } (14),(15),(16), \\[3mm] C(H^{\ell+\mu} + (\dfrac{h}{\varepsilon})^{2q} + \delta + \dfrac{\varepsilon}{\delta}), & \text{if } W(K_{\delta_j}) = H^1_0(K_{\delta_j}) \ (\delta > \varepsilon), \end{cases}$$

where we also assume $\delta \leq r_0$ or $\delta + \varepsilon/\delta \leq r_0$ in the first and third cases, respectively. We use the notation $H^0(\Omega) = L^2(\Omega)$. The constants C are independent of $H, h, \varepsilon, \delta$.

If in addition to the assumptions of Theorem 4.1, the map $s \in \mathbb{R} \mapsto a^\varepsilon(\cdot, s) \in (W^{1,\infty}(\Omega))^d$ is of class C^2 with first and second derivatives bounded by $C\varepsilon^{-1}$, then for sufficiently fine meshes and modeling errors, (e.g. in the second case of Theorem 4.1, for $(h/\varepsilon)^{2q} \leq H \leq H_1$) one can show the convergence of a Newton method, and the uniqueness of the numerical solution u^H.

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