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Marco Discacciati1, Paola Gervasio2, and Alfio Quarteroni3,4

1Laboratori de Càlcul Numèric (LaCaN), Escola Tècnica Superior d’Enginyers de Camins, Canals i Ports de Barcelona (ETSECCPB), Universitat Politècnica de Catalunya (UPC BarcelonaTech), Campus Nord UPC - C2, E-08034 Barcelona, Spain
2DICATAM, Università di Brescia, via Branze 38, I-25123 Brescia, Italy
3MOX, Politecnico di Milano, Piazza Leonardo da Vinci 32, I-20133 Milano, Italy and MATHICSE, Chair of Modelling and Scientific Computing, Ecole Polytechnique Fédérale de Lausanne, Station 8, CH-1015 Lausanne, Switzerland

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We study the Interface Control Domain Decomposition (ICDD) for the Stokes equation. We reformulate this problem introducing auxiliary control variables that represent either the traces of the fluid velocity or the normal stress across subdomain interfaces. Then, we characterize suitable cost functionals whose minimization permits to recover the solution of the original problem. We analyze the well-posedness of the optimal control problems associated to the different choices of the cost functionals, and we propose a discretization of the problem based on hp finite elements. The effectiveness of the proposed methods is illustrated through several numerical tests.

Keywords: Stokes equations, Domain Decomposition Methods, Optimal control, hp-Finite Elements, ICDD

1. INTRODUCTION

The Interface Control Domain Decomposition (ICDD) method was introduced in [4,5] as a solution strategy for boundary value problems governed by elliptic partial differential equations. In this paper we extend this methodology to the Stokes equations and we study its effectiveness in computing the solution of this linear model for laminar incompressible flows.

The ICDD method, which shares some similarities with the classic overlapping Schwarz method [17–19] and with the Least Square Conjugate Gradient [10] and the Virtual Control [13] methods, is characterized by a decomposition of the original domain into overlapping regions and by the introduction of new auxiliary variables on the subdomain interfaces. In the case of the Stokes problem, these variables may represent either the trace of the fluid velocity or the normal stress across the interfaces. In either case, they play the role of control variables that can be determined as solution of an optimal control problem that imposes the minimization of a suitably defined cost functional involving the solutions of well-posed local subproblems.

The ICDD method can thus be regarded as a novel domain decomposition method whose interest lies in the fact that, at least in the case of two subdomains, it may show convergence rates independent of the computational grid, of the polynomial degree used for the numerical approximation and, for a particular choice of the cost functional, also independence on the size of the overlapping. The choice of the cost functional is crucial to ensure the uniqueness of the solution on the overlapping area. In particular, we show that, for the Stokes problem, the cost functionals must account for both the velocity and the pressure across the interfaces to ensure the matching of these two variables in the overlapping regions.

What makes the ICDD method even more attractive is also its capability of handling differential problems of heterogeneous type, i.e., governed by different type of equations in different subregions of the computational domain. Some examples of such application of the method were provided in [4,5] in the case of advection/advection-diffusion problems. Another interesting problem with many significant applications is the coupling of Stokes and Darcy equations to model filtration processes (see [3,4,6,14]).

The outline of the paper is as follows. In section 2 we write the Stokes problem in a bounded domain and we reformulate it in equivalent ways after splitting the original domain into two overlapping regions. In section 3 we introduce a discretization of the problem using hp finite elements, we present the ICDD method considering the cases of Dirichlet, Neumann and mixed control variables. In each case we write the corresponding optimality system with its algebraic counterpart. In section 4 we present several numerical results aimed at studying the convergence behavior of the proposed ICDD methods with respect to the grid size, the polynomial degree, and the size of the overlapping region. Finally, section 5 is devoted to the theoretical analysis of the different methods.

2. PROBLEM SETTING

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be an open bounded domain with Lipschitz boundary $\partial \Omega$. We assume that $\partial \Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$ and that $\Gamma_D \neq \emptyset$ while $\Gamma_N$ might be empty. We consider the Stokes problem:
Problem $\mathcal{P}_\Omega$:

\begin{align}
-\text{div} \, T(u, p) & = f \quad \text{in } \Omega \\
\text{div} \, u & = 0 \quad \text{in } \Omega \\
u & = \phi_D \quad \text{on } \Gamma_D \\
T(u, p) \cdot n & = \phi_N \quad \text{on } \Gamma_N
\end{align}

(1)

describing the motion of a steady, viscous, incompressible fluid confined in the region $\Omega$. Here, $T(u, p) = 2\nu \nabla u - pI$ is the Cauchy stress tensor being $\nabla u = \frac{1}{2}(\nabla u + (\nabla u)^T)$, $\nu > 0$ is the fluid viscosity, $u$ is its velocity and $p$ its pressure and $n$ is the unit normal vector to $\partial \Omega$ directed outwards the domain $\Omega$. We assume that $f \in [L^2(\Omega)]^d$, $\phi_D \in [H^{1/2}(\Gamma_D)]^d$ and $\phi_N \in [H^{-1/2}(\Gamma_N)]^d$ are assigned functions. If $\partial \Omega \equiv \Gamma_D$ (i.e., $\Gamma_N = \emptyset$), the compatibility condition $\int_{\partial \Omega} \phi_D \cdot n = 0$ must hold, and a further condition on $p$, e.g.,

$$\int_{\Omega} p = 0$$

must be enforced to guarantee the well-posedness of problem (1).

The weak form of problem (1) is: find $u \in [H^1(\Omega)]^d$, $p \in L^2(\Omega)$ such that, for all $v \in [H^1(\Omega)]^d$, $q \in L^2(\Omega)$,

$$a(u, v) + b(p, v) = \int_{\Omega} f \cdot v + \int_{\Gamma_N} \phi_N \cdot v$$

$$b(q, u) = 0,$$

where

$$a(u, v) = \int_{\Omega} \nu (\nabla u + (\nabla u)^T) : \nabla v$$

(2)

and

$$b(q, v) = -\int_{\Omega} q \text{div} v.$$ 

(3)

For simplicity of exposition, in the rest of the paper we will often use the strong form of the Stokes problem, but it must be understood that the analysis is carried out in the weak setting.

We consider an overlapping decomposition of the domain $\Omega$ in two subdomains $\Omega_1$ and $\Omega_2$: $\Omega = \Omega_1 \cup \Omega_2$. We denote the overlapping region by $\Omega_{12} = \Omega_1 \cap \Omega_2$ and let $\Gamma_i = \partial \Omega_i \setminus \partial \Omega$. Moreover, let $\Gamma_D = \Gamma_D \cap \partial \Omega_1$ and $\Gamma_N = \Gamma_N \cap \partial \Omega_1$ (see figure 1).

We reformulate the Stokes problem (1) on the split domain in the following possible ways.

**Problem $\mathcal{P}_{\Gamma,i}$**:

\begin{align}
-\text{div} \, T(u_i, p_i) & = f \quad \text{in } \Omega_i, \ i = 1, 2 \\
\text{div} \, u_i & = 0 \quad \text{in } \Omega_i, \ i = 1, 2 \\
u_i & = \phi_D \quad \text{on } \Gamma_D, \ i = 1, 2 \\
T(u_i, p_i) \cdot n & = \phi_N \quad \text{on } \Gamma_N, \ i = 1, 2 \\
u_i & = u_2 \quad \text{on } \Gamma_1 \cup \Gamma_2.
\end{align}

(4)

FIG. 1: Representation of the computational domain $\Omega$ and of its overlapping splitting.

In case $\Gamma_N^i = \emptyset$ for some $i$, we would supplement (4) with the condition

$$\int_{\Omega_i} p_i = 0$$

to ensure the well-posedness of the corresponding local problem.

**Problem $\mathcal{P}_{\Gamma,\Omega}$**:

\begin{align}
-\text{div} \, T(u_1, p_1) & = f \quad \text{in } \Omega_1, \ i = 1, 2 \\
\text{div} \, u_1 & = 0 \quad \text{in } \Omega_1, \ i = 1, 2 \\
u_1 & = \phi_D \quad \text{on } \Gamma_D, \ i = 1, 2 \\
T(u_1, p_1) \cdot n & = \phi_N \quad \text{on } \Gamma_N, \ i = 1, 2 \\
u_1 & = u_2 \quad \text{on } \Gamma_1, \\
T(u_1, p_1) \cdot n & = T(u_2, p_2) \cdot n \quad \text{on } \Omega_{12}.
\end{align}

(5)

Condition (5) on $\Gamma_1$ should be understood as follows. The normal vector $n$ on $\Gamma_1$ is directed outward of $\Omega_1$ and the normal component of the tensor $T(u_2, p_2)$ is computed upon restricting it to $\Omega_{12}$. On the other hand, on $\Gamma_2$ the normal vector $n$ is directed outward of $\Omega_2$ and the normal component of the tensor $T(u_1, p_1)$ is taken upon restricting it to $\Omega_{12}$.

Moreover, we consider the problem:

**Problem $\mathcal{P}_{\Gamma,\Omega}$**:

\begin{align}
-\text{div} \, T(u_1, p_1) & = f \quad \text{in } \Omega_1, \ i = 1, 2 \\
\text{div} \, u_1 & = 0 \quad \text{in } \Omega_1, \ i = 1, 2 \\
u_1 & = \phi_D \quad \text{on } \Gamma_D, \ i = 1, 2 \\
T(u_1, p_1) \cdot n & = \phi_N \quad \text{on } \Gamma_N, \ i = 1, 2 \\
u_1 & = u_2 \quad \text{on } \Gamma_1, \\
T(u_1, p_1) \cdot n & = T(u_2, p_2) \cdot n \quad \text{on } \Omega_{12}.
\end{align}

(6)

If $\Gamma_N^i = \emptyset$, we should impose

$$\int_{\Omega_i} p_i = 0$$

to guarantee the well-posedness of the Stokes problem in $\Omega_1$. 

Let us introduce the following spaces

$$V = [H^1(\Omega)]^d, \quad V_i = [H^1(\Omega_i)]^d, \quad i = 1, 2$$

$$Q = L^2(\Omega), \quad Q_0 = \{q \in Q : \int_\Omega q = 0\}$$

$$Q_i = L^2(\Omega_i), \quad Q_{i,0} = \{q \in Q_i : \int_{\Omega_i} q = 0\} \quad i = 1, 2$$

and the following affine manifolds

$$V_{\phi_D} = \{v \in [H^1(\Omega)]^d : v = \phi_D on \Gamma_D\}$$

$$V_{i,\phi_D} = \{v \in [H^1(\Omega_i)]^d : v = \phi_D on \Gamma_{i,D}\}, \quad i = 1, 2.$$  

Finally, we set

$$V_{i,0} = \{v \in [H^1(\Omega_i)]^d : v = 0 on \Gamma_{i,D}\}, \quad i = 1, 2.$$  

To prove that the Stokes problem (1) is equivalent to either (3), or (5), or (6), we will denote $w = u_{1|\Omega_1} - u_{2|\Omega_2}$ and $q = p_1|\Omega_1 - p_2|\Omega_2$ the difference in $\Omega_{12}$ between the local solutions. Note that $(w, q)$ satisfies the Stokes equations:

$$\begin{align*}
-\text{div} \nabla (w, q) &= 0 \quad \text{in } \Omega_{12} \\
\text{div} w &= 0 \quad \text{in } \Omega_{12}.
\end{align*}$$

The boundary conditions fulfilled by $w$ and $q$ on $\partial\Omega_{12}$, as well as the spaces to which these functions belong will be specified case by case.

**Assumption 2.1.** We suppose that one of the following assumptions is verified: $\Gamma_N = \emptyset; \Gamma_N \neq \emptyset$ and $\Gamma_N \cap \partial\Omega_{12} \neq \emptyset; \Gamma_N \cap \partial\Omega_{12} = \emptyset$ with $\Gamma_N \neq \emptyset$ connected.

**Proposition 2.1 (Equivalence between $P_\Omega$ and $P_{\Gamma,t}$).** The Stokes problems $P_\Omega$ and $P_{\Gamma,t}$ are equivalent if Assumption 2.1 holds. Equivalence holds in the sense that if $(u, p) \in V_{\phi_D} \times Q$ and $(u, p_i) (i = 1, 2)$ are the unique solutions of $P_\Omega$ and $P_{\Gamma,t}$ respectively, there exist two uniquely determined constants $C_1, C_2 \in \mathbb{R}$, possibly null, such that, for $i = 1, 2,$ $u_{i|\Omega_i} = u_i$ and $p_{i|\Omega_i} = p_i + C_i.$

**Proof.** We treat the different cases separately.

1. Assume first that $\Gamma_N \cap \partial\Omega_{12} \neq \emptyset.$ Then, problem (1) is well-posed in $(u, p) \in V_{\phi_D} \times Q$ and the restrictions of its solution to $\Omega_i$ satisfy (4) by construction.

Viceversa, for $i = 1, 2$, let $(u, p_i) \in V_{i,\phi_D} \times Q_i$, $(i = 1, 2)$ be the solutions of the well-posed local problems

$$\begin{align*}
-\text{div} \nabla (u_i, p_i) &= f \quad \text{in } \Omega_i \\
\text{div} u_i &= 0 \quad \text{in } \Omega_i \\
u_i &= \phi_D on \Gamma_{i,D}
\end{align*}$$

$$\begin{align*}
T(u_i, p_i) = &\phi_N on \Gamma_{i,N} \\
u_i = u_j on \Gamma_{i,j}, \quad j = 3 - i.
\end{align*}$$

By construction, the functions $w$ and $q$ satisfy problem (10) with boundary conditions

$$\begin{align*}
T(w, q) &= 0 \quad \text{on } \partial\Omega_{12} \cap \Gamma_N \\
w &= 0 \quad \text{on } \partial\Omega_{12} \setminus \Gamma_N.
\end{align*}$$

This problem is well-posed and admits the unique solution $w = 0$ and $q = 0$, hence $u_1 = u_2$ and $p_1 = p_2$ in $\Omega_{12}$. Thus, we can set

$$u = \begin{cases}
  u_1 & \text{in } \Omega_1 \setminus \Omega_{12} \\
  u_1 = u_2 & \text{in } \Omega_{12}, \\
  u_2 & \text{in } \Omega_2 \setminus \Omega_{12},
\end{cases}$$

and

$$p = \begin{cases}
  p_1 & \text{in } \Omega_1 \setminus \Omega_{12} \\
  p_1 = p_2 & \text{in } \Omega_{12}, \\
  p_2 & \text{in } \Omega_2 \setminus \Omega_{12}.
\end{cases}$$

By construction, functions $u$ and $p$ belong to $V_{\phi_D} \times Q$ and they satisfy problem (1). In case $C_1 = C_2 = 0$.

2. Let now $\Gamma_N \cap \partial\Omega_{12} = \emptyset$ and assume that $\Gamma_N$ is connected. In this case, either $\Gamma_N^1 = \emptyset$ or $\Gamma_N^2 = \emptyset$. We consider the latter case; the former can be treated analogously.

If $(u, p) \in V_{\phi_D} \times Q$ is the solution of $P_{\Gamma,t}$, if we set $u_i = u_{i|\Omega_i} (i = 1, 2)$, $p_i = p_{i|\Omega_i},$

$$p_2 = p_{2|\Omega_2} - \frac{1}{|\Omega_{12}|} \int_{\Omega_2} p_{2|\Omega_2},$$

we can immediately verify that $(u, p_i) \in V_{i,\phi_D} \times Q_i$ $(i = 1, 2)$ are solutions of $P_{\Gamma,t}$ with $\int_{\Omega_2} p_2 = 0$.

Thus, $C_1 = 0$ and $C_2 = -\frac{1}{|\Omega_{12}|} \int_{\Omega_2} p_{2|\Omega_2}.$

Viceversa, let $(u_{1, p_i}) \in V_{1,\phi_D} \times Q_1$, $(u_{2, p_2}) \in V_{2,\phi_D} \times Q_2$ be the solutions of $P_{\Gamma,t}$. The functions $(w, q)$ satisfy (10) with $w = 0$ on $\partial\Omega_{12}$. Then, $w = 0$ and $q = const in \Omega_{12}$. The function $q$ is uniquely determined by $\int_{\Omega_{12}} q = \int_{\Omega_{12}} (p_1 - p_2)$ which implies

$$q = \frac{1}{|\Omega_{12}|} \int_{\Omega_{12}} (p_1 - p_2).$$

If we take $u$ as in (11) and

$$p = \begin{cases}
  p_1 & \text{in } \Omega_1 \setminus \Omega_{12} \\
  p_1 = p_2 + q & \text{in } \Omega_{12}, \\
  p_2 & \text{in } \Omega_2 \setminus \Omega_{12},
\end{cases}$$

then $(u, p)$ satisfy $P_{\Omega}$ and the thesis follows with $C_1 = 0$ and $C_2 = q$.

3. Let $(u, p) \in V_{\phi_D} \times Q_0$ be the solution of $P_{\Omega}$. Then, for $i = 1, 2$, the functions

$$u_i = u_{i|\Omega_i}, \quad p_i = p_{i|\Omega_i} - \frac{1}{|\Omega_{12}|} \int_{\Omega_{12}} p_{2|\Omega_2},$$

belong to $V_{i,\phi_D} \times Q_i$ and they satisfy $P_{\Gamma,t}$. Thus, $C_1 = 0$ and $C_2 = -\frac{1}{|\Omega_{12}|} \int_{\Omega_{12}} p_{2|\Omega_2}.$
Problems and as given by $P$ problems and $w$ defined respectively as in (10) and as in (11) and as

$$q = \frac{1}{|\Omega_{12}|} \int_{\Omega_{12}} (p_1 - p_2).$$

If we define the constants

$$C_1 = \frac{1}{|\Omega|} \left( \int_{\Omega_{12}} p_2 - |\Omega_2 \setminus \Omega_2 q| \right)$$

and

$$C_2 = \frac{1}{|\Omega|} \left( \int_{\Omega_{12}} p_2 + |\Omega_1 q| \right),$$

since $C_2 - C_1 = q$, then $p_1 + C_1 = p_2 + C_2$ in $\Omega_{12}$. Thus, we can easily verify that the functions $u$ and $p$ defined respectively as in (11), and as

$$p = \begin{cases} 
  p_1 + C_1 & \text{in } \Omega \setminus \Omega_1 \\
  p_1 + C_1 = p_2 + C_2 & \text{in } \Omega_{12} \\
  p_2 + C_2 & \text{in } \Omega_2 \setminus \Omega_2 
\end{cases}$$

are solutions of $P_{\Omega}$ with $\int_{\Omega} p = 0$. □

Remark 2.1 If $\partial \Omega_{12} \cap \Gamma_N = \emptyset$ and $\Gamma_N \neq \emptyset$ (i = 1, 2), problems $P_{\Omega}$ and $P_{\Gamma,f}$ are not equivalent.

In fact, if $(u_i, p_i)$ are the solutions of $P_{\Gamma,f}$, the functions $w$ and $q$ satisfy (10) with boundary condition $w = 0$ on $\partial \Omega_{12}$. Then, $w = 0$ and $q = \text{const}$ in $\Omega_{12}$ with $q$ uniquely given by

$$q = \frac{1}{|\Omega_{12}|} \int_{\Omega_{12}} (p_1 - p_2).$$

Then, proceeding similarly to the third case of the proof of Proposition 2.1 there exist two unique constants $C_1, C_2$ with $q = C_2 - C_1$ so that we can define $u$ and $p$ as in (11) and (12), respectively. The Neumann boundary conditions in $P_{\Gamma,f}$ imply

$$T(u_i, p_i) \cdot n = \phi_N \quad \text{on } \Gamma_N,$$

and, by definition of $u$ and $p$, we have

$$T(u, p) \cdot n = \phi_N + C_1 n \quad \text{on } \Gamma_N.$$

Thus, $(u, p)$ satisfy problem $P_{\Omega}$ if and only if $C_1 = C_2 = 0$, but we cannot guarantee that this condition is fulfilled.

Proposition 2.2 (Equivalence between $P_{\Omega}$ and $P_{\Gamma,f}$) If $\partial \Omega_{12} \cap \Gamma_D \neq \emptyset$, the Stokes problems $P_{\Omega}$ and $P_{\Gamma,f}$ are equivalent in the sense that there exist unique constants $C_1, C_2 \in \mathbb{R}$ such that $u_i, \Omega_i = u_i$ and $p_i, \Omega_i = p_i + C_i$, $(u, p)$ and $(u_i, p_i)$ (i = 1, 2) being, respectively, the unique solutions of $P_{\Omega}$ and $P_{\Gamma,f}$.

Proof. The proof goes along the same arguments used for Proposition 2.1 so that we only define the constants in the cases $\Gamma_N \neq \emptyset$ or $\Gamma_N = \emptyset$.

In the first case it is straightforward to see that equivalence holds with $C_1 = C_2 = 0$. On the other hand, if $\Gamma_N = \emptyset$, the functions $w$ and $q$ satisfy the problem (10) with boundary conditions

$$w = 0 \quad \text{on } \partial \Omega_{12} \cap \partial \Omega,$$

and $p = \tilde{p} + C_1$ to recover the null average. □

Remark 2.2 Problems $P_{\Omega}$ and $P_{\Gamma,f}$ are not equivalent if $\partial \Omega_{12} \cap \Gamma_D = \emptyset$. In fact, in this case problem (10) in $\Omega_{12}$ would be supplemented with the boundary condition $T(w, q) \cdot n = 0$ on $\Omega_{12}$ which has infinite non-trivial solutions that may differ one from another not only by a constant.

Proposition 2.3 (Equivalence between $P_{\Omega}$ and $P_{\Gamma,f}$) The Stokes problems $P_{\Omega}$ and $P_{\Gamma,f}$ are equivalent if either $\Gamma_N = \emptyset$, or $\Gamma_N \cap \partial \Omega_{12} \neq \emptyset$, or $\Gamma_N \cap \partial \Omega_{12} = \emptyset$ and $\Gamma_N \neq \emptyset$. Equivalence holds in the sense that if $(u, p)$ and $(u_i, p_i)$ (i = 1, 2) are the unique solutions of $P_{\Omega}$ and $P_{\Gamma,f}$, respectively, there exist two uniquely determined constants $C_1, C_2 \in \mathbb{R}$, possibly null, such that, for $i = 1, 2$, $u_i, \Omega_i = u_i$ and $p_i, \Omega_i = p_i + C_i$.

Proof. The proof develops along the lines of the previous propositions. Let us only point out that equivalence holds with $C_1 = C_2 = 0$ if $\Gamma_N \neq \emptyset$. Otherwise, if $\Gamma_N = \emptyset$, if $(u, p) \in V_{\phi_D} \times Q_0$ is the solution of $P_{\Omega}$, then $u_i = u_i, \Omega_i$ and $p_2 = p_{\Omega_2} + C_i$.

Thus, $(u, p)$ satisfy problem $P_{\Omega}$ if and only if $C_1 = C_2 = 0$, but we cannot guarantee that this condition is fulfilled.
Remark 2.3 Problems $\mathcal{P}_\Gamma$ and $\mathcal{P}_{\Gamma,lf}$ are not equivalent if $\partial \Omega_{12} \cap \Gamma_N = \emptyset$, $\Gamma_N' = \emptyset$, and $\Gamma_0^2 \neq \emptyset$. In fact, if \((u_1, V_1, q_1) \in V_{1,\phi_D} \times Q_{1,0} \) and \((u_2, q) \in V_{2,\phi_D} \times Q_2\) are the solutions of $\mathcal{P}_{\Gamma,lf}$, then \((w, q)\) satisfy problem (10) in $\Omega_{12}$ with boundary condition $T(w, q) \cdot n = 0$ on $\Gamma_2$ and $w = 0$ on $\partial \Omega_{12} \setminus \Gamma_2$. The solution of this problem in $\Omega_{12}$ is identically null. However, since $\int_{\Omega_{12}} q = \int_{\Omega_{12}} (p_1 - p_2)$ with $p_1 \in Q_{1,0}$ and $p_2$ uniquely determined by the Neumann boundary condition on $\Gamma_N^2$, we cannot guarantee that $q = 0$.

Notice that a result similar to Proposition 2.3 could be obtained by switching the role of the interface conditions \(\eta_5\) and \(\eta_6\), i.e., considering

Problem $\mathcal{P}_{\Gamma,fl}$:

$$
\begin{align*}
-\text{div} \, T(u_1, p_1) &= f & \text{in } \Omega_i, \ i = 1, 2, \\
\text{div} \, u_1 &= 0 & \text{in } \Omega_i, \ i = 1, 2, \\
u_1 &= \phi_D & \text{on } \Gamma_D^i, \ i = 1, 2, \\
T(u_1, p_1) \cdot n &= \phi_N & \text{on } \Gamma_N^i, \ i = 1, 2, \\
u_1 &= u_2 & \text{on } \Gamma_2.
\end{align*}
$$

(14)

3. FORMULATION OF THE ICDD METHOD FOR THE STOKES PROBLEM

For the sake of simplicity, we will consider homogeneous boundary conditions, i.e., we will set $\phi_D = 0$ on $\Gamma_D$ and $\phi_N = 0$ on $\Gamma_N$. Moreover, since we will be interested in computing a finite dimensional approximation of the solution of the Stokes problem, we introduce the ICDD method directly at the discrete level.

3.1. $hp$-FEM discretization

We introduce two regular computational grids $T_1$ and $T_2$ in $\Omega_1$ and $\Omega_2$ made by either simplices or quadrilaterals/hexahedra. We suppose that each element $T \in T_i$ is obtained by a $C^1$ diffeomorphism $F_T$ of the reference element $\hat{T}$ and we suppose that two adjacent elements of $T_i$ share either a common vertex or a complete edge or a complete face (when $d = 3$). Moreover, we assume that they coincide in $\Omega_{12}$ and that both interfaces $\Gamma_1$ and $\Gamma_2$ do not cross any element of $\Omega_1$ or $\Omega_2$. We discretize both primal and dual problems in each subdomain by $hp$ finite element methods ($hp$-FEM). Because of the difficulty to compute integrals exactly for large $p$, typically when quadrilaterals are used, Legendre-Gauss-Lobatto quadrature formulas are employed to approximate the bilinear forms $a_{\Gamma_0}$ and $b_{\Omega_0}$ (see [2]-[3]) as well as the $L^2$-inner products in $\Omega_i$ and on the interfaces. This leads to the so called Galerkin approach with Numerical Integration (G-NI) [1,2] and to the Spectral Element Method with Numerical Integration (SEM-NI).

In particular, we consider either inf-sup stable finite dimensional spaces or stabilized couples of spaces of the same degree (see [7,8,11,15]) to approximate the velocity and the pressure and we assume that the polynomials used for the pressure are continuous (see, e.g., [16]). More precisely, given an integer $p \geq 1$, let $P_p$ be the space of polynomials whose global degree is less than or equal to $p$ in the variables $x_1, \ldots, x_d$ and $Q_p$ be the space of polynomials that are of degree less than or equal to $p$ with respect to each variable $x_1, \ldots, x_d$. The space $P_p$ is associated to simplicial partitions, while $Q_p$ to quadrilateral ones. We introduce the finite dimensional space on $\Omega_i$ defined by

$$
X_{i, h}^P = \{ v \in C^0(\Omega_i) : v\rvert_T \in P_p, \forall T \in T_i \}
$$

in the simplicial case, and by

$$
X_{i, h}^P = \{ v \in C^0(\Omega_i) : v\rvert_T \circ F_T \in Q_p, \forall T \in T_i \}
$$

for quadrilaterals. Then, the finite dimensional spaces for velocity and pressure are, respectively,

$$
V_{i, h} = [X_{i, h}^P]^d \cap V_{i, 0}, \quad Q_{i, h} = X_{i, h}^P
$$

(15)

for suitable polynomial degrees $p$ and $r$.

3.2. ICDD method with Dirichlet controls

Assume, for simplicity, that $\partial \Omega_{12} \cap \Gamma_N \neq \emptyset$ and $\Gamma_D \neq \emptyset$. (We will discuss this issue more in details in section 5) We define the space of discrete Dirichlet controls as

$$
\Lambda_{i, h} = \{ \lambda_{i, h} \in C^0(\Gamma_i) : \exists v_{i, h} \in V_{i, h} \text{ with } \lambda_{i, h} = v_{i, h}\rvert_{\Gamma_i} \}.
$$

and let

$$
\Lambda_{i, h}^D = \Lambda_{i, h}^D \times \Lambda_{i, h}^2.
$$

For $i = 1, 2$, we consider two control functions $\lambda_{i, h} \in \Lambda_{i, h}$ and the state problems: find $(u_{i, h}, p_{i, h}) \in V_{i, h} \times Q_{i, h}$ such that, for all $(v_{i, h}, q_{i, h}) \in V_{i, h} \times Q_{i, h}$, $v_{i, h} = 0$ on $\Gamma_i$,

$$
\begin{align*}
a_i(u_{i, h}, v_{i, h}) + b_i(p_{i, h}, v_{i, h}) &= \int_{\Omega_i} f \cdot v_{i, h} \\
b_i(q_{i, h}, u_{i, h}) &= 0 \\
u_{i, h} &= \lambda_{i, h} \text{ on } \Gamma_i
\end{align*}
$$

(16)

where $a_i$ and $b_i$ denote the restriction of the bilinear forms $a_{\Omega_i}$ and $b_{\Omega_i}$ to $\Omega_i$ and $\Gamma_i$. In fact, $(u_{i, h}, p_{i, h})$ depends on both $\lambda_{i, h}$ and $f$, however such dependence will be understood for the sake of notation.

The unknown controls on the interface are obtained by solving a minimization problem for a cost functional suitably depending on the difference between $u_{1, h}$ and $u_{2, h}$ on the interfaces $\Gamma_1$ and $\Gamma_2$. More precisely, inspired by [2]-[3], we look for

$$
\Delta = (\lambda_{1, h}, \lambda_{2, h}) := \inf \left[ J_{\Delta}(\Delta) := \frac{1}{2}\| u_{1, h} - u_{2, h} \|^2_{L^2(\Gamma_1 \cup \Gamma_2)} \right].
$$

(17)
To the minimization problem \( \min x \), we can associate the following optimality system: find \( \lambda_{i,j} \in A_{h,j}^D \) and, for \( i = 1, 2 \), 
\[
\begin{align*}
\sum_i (u_{i,h}, v_{i,h}) + b_i(p_{i,h}, v_{i,h}) &= \int_{\Omega_i} f \cdot v_{i,h} \\
\sum_i (p_{i,h}, v_{i,h}) &= 0
\end{align*}
\] (18)

where \( \Omega_i \) is the domain of the variational problem in \( \Gamma_i \), and for \( i = 1, 2 \), 
\[
\begin{align*}
\sum_i (w_{i,h}, v_{i,h}) + b_i(q_{i,h}, v_{i,h}) &= 0 \\
\sum_i (q_{i,h}, v_{i,h}) &= 0
\end{align*}
\] (19)

and, for all \( (\mu_{1,h}, \mu_{2,h}) \in A_{h}^D \), 
\[
\int_{\Gamma_1} ((u_{1,h} - u_{2,h}) + w_{1,h}) d\Gamma = 0
\] (20)

3.3. Algebraic formulation of ICDD with Dirichlet controls

To the Stokes problem in subdomain \( \Omega_i \) \( (i = 1, 2) \) we can associate the matrix
\[
S_i = \begin{pmatrix}
A_i & B_i^T \\
B_i & 0
\end{pmatrix}
\]

where \( A_i \) corresponds to the finite dimensional approximation of the bilinear form \( a_{i,\Omega_i} \) (see (2)), while \( B_i \) corresponds to the discretization of \( b_{i,\Omega_i} \) (see (3)). When stabilization is used, the matrices \( S_i \) take the form
\[
S_i = \begin{pmatrix}
A_i & B_i^T \\
B_i & 0
\end{pmatrix} + \begin{pmatrix}
\tilde{A}_i & \tilde{B}_i^T \\
\tilde{B}_i & \tilde{C}_i
\end{pmatrix}
\]

where \( \tilde{A}_i, \tilde{B}_i \) and \( \tilde{C}_i \) are assembled locally, element by element, and they take into account the integration of the differential Stokes operator.

In the following we will denote by the index \( I_i \) the degrees of freedom for the velocity and the pressure belonging to \( \Omega_i \setminus \Gamma_i \), while the index \( \Gamma_i \) will refer to the degrees of freedom on the interface \( \Gamma_i \). For the sake of exposition, we will reorder the nodes in \( \Omega_i \) putting those associated to \( \Omega_i \setminus \Gamma_i \) first followed by those on the interfaces. Correspondingly, with obvious choice of notation, we can rewrite the Stokes matrix \( S_i \) as
\[
S_i = \begin{pmatrix}
\begin{pmatrix}
A_{I_i, I_i} & B_{I_i, I_i}^T \\
B_{I_i, I_i} & 0
\end{pmatrix} & \begin{pmatrix}
A_{I_i, \Gamma_i} & B_{I_i, \Gamma_i}^T \\
B_{I_i, \Gamma_i} & 0
\end{pmatrix} \\
\begin{pmatrix}
A_{\Gamma_i, I_i} & B_{\Gamma_i, I_i}^T \\
B_{\Gamma_i, I_i} & 0
\end{pmatrix} & \begin{pmatrix}
A_{\Gamma_i, \Gamma_i} & B_{\Gamma_i, \Gamma_i}^T \\
B_{\Gamma_i, \Gamma_i} & 0
\end{pmatrix}
\end{pmatrix}
\]

Moreover, we will indicate by \( M_{\Gamma_i} \) the mass matrix on the interface \( \Gamma_i \).

Finally, in the rest of the section, we will denote by \( F_i \) the right-hand side for the state problems in \( \Omega_i \), while \( U_i \) and \( W_i \) will be the vectors of unknown velocity and pressure in \( \Omega_i \) for the state and the adjoint problems, respectively. \( \lambda_{\Gamma_i} \) is the vector of the unknown Dirichlet controls on \( \Gamma_i \):
\[
\begin{pmatrix}
\lambda_{\Gamma_i} \in N_{\Gamma_i} \end{pmatrix}
\]

where \( G_i \) is the set of the elements \( N_{\Gamma_i} \) indices corresponding to the velocity degrees of freedom on the interface \( \Gamma_i \) and \( \chi_j \) is a node on \( \Gamma_i \) \((\lambda_{\Gamma_i})_j \) is the nodal value of the discrete control function \( \lambda_{i,h} \) at the node \( \chi_j \).

We consider now the optimality system associated to the functional \( J_i \) with Dirichlet controls that we introduced in section \( 5.1.5.1 \). If \( R_{ij} \) denotes the algebraic restriction operator of the velocity unknowns in \( \Omega_j \) to the interface \( \Gamma_i \) \((i, j = 1, 2) \), the algebraic counterpart of \( (58)-(60) \) reads:
\[
S_i Y_i = b_i
\]

where \( y_i = (U_{I_i}, U_{\Gamma_i}, W_{I_i}, W_{\Gamma_i}, \lambda_{\Gamma_i}, \lambda_{\Gamma_i})^T \) and the matrix \( S_i \) is defined as
\[
\begin{pmatrix}
S_{I_i, I_i} & 0 & 0 & S_{I_i, \Gamma_i} & 0 & 0 \\
0 & S_{\Gamma_i, \Gamma_i} & 0 & 0 & 0 & 0 \\
0 & 0 & S_{\Gamma_i, \Gamma_i} & 0 & 0 & 0 \\
-S_{\Gamma_i, \Gamma_i} R_{I_i, \Gamma_i} & 0 & 0 & S_{\Gamma_i, \Gamma_i} & 0 & 0 \\
0 & -M_{\Gamma_i, \Gamma_i} & 0 & 0 & M_{\Gamma_i, \Gamma_i} & 0 \\
0 & 0 & M_{\Gamma_i, \Gamma_i} & 0 & 0 & M_{\Gamma_i, \Gamma_i}
\end{pmatrix}
\]

For the numerical solution of the linear systems \( (21) \), we compute the Schur complement system with respect to the control variables \( \lambda_{\Gamma_1}, \lambda_{\Gamma_2} \) and solve them through an iterative method like, e.g., Bi-CGSTAB \( (21) \).

The Schur complement system reads
\[
\Sigma_i \begin{pmatrix}
\lambda_{\Gamma_1} \\
\lambda_{\Gamma_2}
\end{pmatrix} = \chi_i
\]

where
\[
\Sigma_i = \begin{pmatrix}
M_{\Gamma_1} (I_{\Gamma_1} - (R_{I_i, I_i} S_{I_i, I_i}^{-1} S_{I_i, \Gamma_i})^2) \\
M_{\Gamma_2} (I_{\Gamma_2} - (R_{I_i, I_i} S_{I_i, I_i}^{-1} S_{I_i, \Gamma_i})^2)
\end{pmatrix}
\]

and
\[
\chi_i = \begin{pmatrix}
M_{\Gamma_1} R_{I_i, \Gamma_i} (I_{\Gamma_i} - S_{I_i, I_i}^{-1} S_{I_i, \Gamma_i} R_{I_i, \Gamma_i}) S_{I_i, I_i}^{-1} F_1 \\
M_{\Gamma_2} R_{I_i, \Gamma_i} (I_{\Gamma_i} - S_{I_i, I_i}^{-1} S_{I_i, \Gamma_i} R_{I_i, \Gamma_i}) S_{I_i, I_i}^{-1} F_2
\end{pmatrix}
\]

\( I_{\Gamma_i} \) is the identity matrix on the interface \( \Gamma_i \).
3.4. ICDD method with Neumann and mixed controls

Let \( \Lambda^N_{i,h} \) denote the space of discrete Neumann controls on \( \Gamma_i \). We require that \( \Lambda^N_{i,h} \subset L^2(\Gamma_i) \).

For \( i = 1, 2 \), given the control functions \( \lambda_{i,h} \in \Lambda^N_{i,h} \) and the discrete state problems: find \( (u_i, p_i, q_i, h_i) \in V_i \times Q_i \), such that, for all \( (v_i, q_i, h_i) \in V_i \times Q_i \),

\[
a_i(u_i, v_i, h_i) + b_i(p_i, v_i, h_i) = \int_{\Gamma_i} \lambda_{i,h} \cdot v_i \\
+ \int_{\Omega} f \cdot v_i, \\
b_i(q_i, u_i, h_i) = 0.
\]  
(23)

Let \( T_k \subset \Omega \) be a generic element in \( \Omega_i \); we introduce the set \( \mathcal{E}_i = \{ k : \text{meas}(\partial T_k \cap \Gamma_i) > 0 \} \) and, for any \( k \in \mathcal{E}_i \), the edges \( e_{ik} = \partial T_k \cap \Gamma_i \). Thanks to the definition of \([X^p_{ik}] \) for any \( v_{ik} \in [X^p_{ik}] \) and \( q_{ik} \in X^q_{ik} \), it holds \( v_{ik} |_{\mathcal{E}_i} \in C^1(\mathcal{E}_i) \) and \( q_{ik} |_{\mathcal{E}_i} \in C^0(\mathcal{E}_i) \) and then we define the discrete normal stress

\[
\Phi_{i,j,h} = T(u_i, h_i, p_i, h_i) \cdot n \quad \text{on} \quad \Gamma_j.
\]

This definition makes sense in classic way on each \( e_{ik} \subset \Gamma_i \), so that \( \Phi_{i,j,h} \in L^2(\Gamma_i) \).

We are interested in evaluating the discrete normal stress associated to \( (u_i, h_i, p_i, h_i) \) also on the interface \( \Gamma_j \) (\( j = 3-i \)), which is internal to \( \Omega_i \).

With this aim we first restrict \( (u_i, h_i, p_i, h_i) \) to \( \Omega_{i,2} \) and then extend it to \( \Omega_j \) in such a way that such extension \( (\tilde{u}_i, \tilde{h}_i, \tilde{p}_i, \tilde{h}_i) \) belongs to \( V_i \times Q_i \). Then we define

\[
\Phi_{i,j,h} = T(\tilde{u}_i, h_i, \tilde{p}_i, h_i) \cdot n \quad \text{on} \quad \Gamma_j
\]

and it holds \( \Phi_{i,j,h} \in [L^2(\Gamma_i)]^d \).

Following (25), the discrete Neumann controls \( \lambda_{i,j,h} \) on the interface \( \Gamma_i \) are obtained as solution of the following minimization problem

\[
\inf_{\lambda_{1,h}, \lambda_{2,h}} \left\{ J_f(\lambda_{1,h}, \lambda_{2,h}) = \frac{1}{2} \sum_{i=1}^2 \left\| \Phi_{i,j,h} - \Phi_{i,h} \right\|^2_{L^2(\Gamma_i)} \right\}.
\]  
(24)

Similarly, for any \( \ell \in \mathcal{G}^N_{i} \) and \( \varphi_{j,\ell} \in \mathcal{B}^N_{j} \) we define the vectors \( (\Phi_{i,j,h})_{\ell} \in \mathbb{R}^d \) of the weak discrete normal stresses on \( \Gamma_j \) associated to \( (u_i, h_i, p_i, h_i) \) as

\[
(\Phi_{i,j,h})_{\ell} = a_i(u_i, \varphi_{i,\ell}, h_i) + \sum_{k \in T_i^p} b_i(p_i, \varphi_{i,\ell}, h_i)
- \int_{\Gamma_i} f \cdot \varphi_{i,\ell}.
\]  
(25)

It holds

\[
\left( \Phi_{i,j,h} \right)_{\ell} = \int_{\Gamma_j} \Phi_{i,j,h} \cdot \varphi_{j,\ell} \quad \forall \ell \in \mathcal{G}^N_{i}, \quad i, j \in \{1, 2\}.
\]

To the minimization of problem (24) we can associate the following optimality system: find \( \lambda_{1,h}, \lambda_{2,h} \in \Lambda^N_{h} \) and, for \( i = 1, 2 \), \( (u_i, h_i, p_i, h_i) \), \( (w_i, h_i, q_i, h_i) \in V_i \times Q_i \) such that

\[
\begin{align*}
\phi_i & = a_i(u_i, \varphi_{i,\ell}, h_i) + \sum_{k \in T_i^p} b_i(p_i, \varphi_{i,\ell}, h_i) \\
& + \int_{\Omega} f \cdot \varphi_{i,\ell}, \\
b_i & = 0.
\end{align*}
\]  
(27)

\[
\begin{align*}
\phi_{i,h} & = a_i(w_i, \varphi_{i,\ell}, h_i) + \sum_{k \in T_i^p} b_i(q_i, \varphi_{i,\ell}, h_i) \\
& = (\Phi_{i,j,h})_{\ell} - (\Phi_{j,j,h})_{\ell} \\
& + \int_{\Omega} f \cdot \varphi_{i,\ell}, \\
b_i & = 0.
\end{align*}
\]  
(28)

and

\[
\sum_{i=1}^2 (\Phi_{i,j,h})_{\ell} - (\Phi_{i,j,h})_{\ell} + (\Phi_{j,j,h})_{\ell} = 0
\]  
(29)

where \( j = 3-i \) and

\[
(\Psi_{i,j,h})_{\ell} = a_i(w_i, \varphi_{i,\ell}, h_i) + \sum_{k \in T_i^p} b_i(q_i, \varphi_{i,\ell}, h_i)
\]

is the weak representation of the discrete normal stress on \( \Gamma_j \) associated to the dual state solution \( (u_i, h_i, q_i, h_i) \).

An alternative strategy consists in choosing mixed controls, e.g., a discrete Dirichlet control \( \lambda_{1,h} \in \Lambda^D_{1,h} \) on \( \Gamma_1 \) and a Neumann control \( \lambda_{2,h} \in \Lambda^N_{2,h} \) on \( \Gamma_2 \) and to minimize the difference between both interface velocities and interface normal stresses.

Following (25), the corresponding minimization problems would read:

\[
\inf_{\lambda_{1,h}, \lambda_{2,h}} \left\{ J_f(\lambda_{1,h}, \lambda_{2,h}) : \right. \\
= \frac{1}{2} \left\| u_{1,h} - u_{2,h} \right\|^2_{L^2(\Gamma_1)} \\
+ \frac{1}{2} \left\| \hat{\Phi}_{1,2,h} - \hat{\Phi}_{2,2,h} \right\|^2_{L^2(\Gamma_2)}
\]  
(30)

Alternatively, following (13) and (14), we could consider a discrete Neumann control on \( \Gamma_1 \) and a discrete
Dirichlet control on $\Gamma_2$ and the corresponding minimization problem:

$$\inf_{\lambda_{1,h}, \lambda_{2,h}} \left[ J_f(\lambda_{1,h}, \lambda_{2,h}) := \frac{1}{2} \| \hat{\phi}_{1,1,h} - \phi_{2,1,h} \|^2_{L^2(\Gamma_1)} \\
+ \frac{1}{2} \| u_{1,h} - u_{2,h} \|^2_{L^2(\Gamma_2)} \right]. \tag{31}$$

To the minimization problem (30) we associate the following optimality system: find $\lambda_{1,h}, \lambda_{2,h} \in A^{1,h} \times A^{2,h}$ and, for $i = 1, 2$, $(u_{i,h}, p_{i,h}) \in V_{i,h} \times Q_{i,h}$, $(w_i, q_i) \in V_{i,h} \times Q_{i,h}$ such that

$$a_1(u_{1,h}, \varphi_{1,\ell}, \chi_{1,h}) + b_1(p_{1,h}, \varphi_{1,\ell}) = \int_{\Omega_1} f \cdot \varphi_{1,\ell}, \forall \ell \in \mathcal{I}_1^u, b_1(\psi_{1,h}, u_{1,h}) = 0 \quad \text{on } \Gamma_1, u_{1,h} = \lambda_{1,h} \quad \text{in } \Omega_1 \tag{32}$$

$$a_2(u_{2,h}, \varphi_{2,\ell}, \chi_{2,h}) + b_2(p_{2,h}, \varphi_{2,\ell}) = \int_{\Omega_2} \lambda_{2,h} \cdot \varphi_{2,\ell} \\
+ \int_{\Omega_2} f \cdot \varphi_{2,\ell}, \forall \ell \in \mathcal{I}_2^u, b_2(\psi_{2,h}, u_{2,h}) = 0 \quad \text{in } \Omega_2 \tag{33}$$

$$a_1(w_{1,h}, \varphi_{1,\ell}, \chi_{1,h}) + b_1(q_{1,h}, \varphi_{1,\ell}) = 0 \quad \forall \ell \in \mathcal{I}_1^u, b_1(\psi_{1,h}, w_{1,h}) = 0 \quad \text{on } \Gamma_1, w_{1,h} = u_{1,h} - u_{2,h} \quad \text{on } \Gamma_1 \tag{34}$$

$$a_2(w_{2,h}, \varphi_{2,\ell}, \chi_{2,h}) + b_2(q_{2,h}, \varphi_{2,\ell}) = (\Phi_{2,G_2}) \ell - (\Phi_{1,G_2}) \ell \quad \forall \ell \in \mathcal{I}_2^u, b_2(\psi_{2,h}, w_{2,h}) = 0 \quad \text{in } \Omega_2 \tag{35}$$

and

$$[(u_{1,h|\Gamma_1}) \ell - (u_{2,h|\Gamma_1}) \ell + (w_{2,h|\Gamma_1}) \ell] \\
+ [(\Phi_{1,G_2}) \ell + (\Phi_{2,G_2}) \ell + (\Psi_{1,G_2}) \ell] = 0 \quad \forall \ell \in \mathcal{I}_1^u, \forall j \in \mathcal{I}_2^u. \tag{36}$$

To the minimization problem (31) we now associate the optimality system: find $\lambda_{1,h}, \lambda_{2,h} \in A^{1,h} \times A^{2,h}$ and, for $i = 1, 2$, $(u_{i,h}, p_{i,h}) \in V_{i,h} \times Q_{i,h}$, $(w_i, q_i) \in V_{i,h} \times Q_{i,h}$ such that

$$a_1(u_{1,h}, \varphi_{1,\ell}, \chi_{1,h}) + b_1(p_{1,h}, \varphi_{1,\ell}) = \int_{\Omega_1} f \cdot \varphi_{1,\ell}, \forall \ell \in \mathcal{I}_1^u, b_1(\psi_{1,h}, u_{1,h}) = 0 \quad \text{in } \Omega_1 \tag{37}$$

$$a_2(u_{2,h}, \varphi_{2,\ell}, \chi_{2,h}) + b_2(p_{2,h}, \varphi_{2,\ell}) = \int_{\Omega_2} f \cdot \varphi_{2,\ell}, \forall \ell \in \mathcal{I}_2^u, b_2(\psi_{2,h}, u_{2,h}) = 0 \quad \text{in } \Omega_2 \tag{38}$$

$$a_1(w_{1,h}, \varphi_{1,\ell}, \chi_{1,h}) + b_1(q_{1,h}, \varphi_{1,\ell}) = (\Phi_{1,G_1}) \ell - (\Phi_{2,G_1}) \ell \quad \forall \ell \in \mathcal{I}_1^u, b_1(\psi_{1,h}, w_{1,h}) = 0 \quad \text{on } \Gamma_2 \tag{39}$$

$$a_2(w_{2,h}, \varphi_{2,\ell}, \chi_{2,h}) + b_2(q_{2,h}, \varphi_{2,\ell}) = 0 \quad \forall \ell \in \mathcal{I}_2^u, b_2(\psi_{2,h}, w_{2,h}) = 0 \quad \text{on } \Gamma_2 \tag{40}$$

$$w_{2,h} = u_{1,h} - u_{2,h} \quad \text{on } \Gamma_2 \tag{41}$$

3.5. Algebraic formulation of ICDD with Neumann and mixed controls

Using the previous notations, the discrete values of the Neumann controls are given by

$$(\lambda_{i,h}) = \sum_{k \in \mathcal{E}_i} \int_{e_k} \chi_{i,h} \cdot \varphi_{i,\ell} \quad \forall \ell \in \mathcal{I}_i^u. \tag{42}$$

Denoting by $T_{ij}$ the finite dimensional counterpart of the operator that associates to the velocity and pressure in $\Omega_i$ the corresponding normal stress tensor on the interface $\Gamma_j$ ($j = 1, 2$) (as in (25)), after discretization the optimality system (27)-(29) for the functional $J_f$ with Neumann controls yields the following matrix:

$$\begin{pmatrix}
S_1 & 0 & 0 & 0 & -I_{G_1} & 0 \\
0 & S_2 & 0 & 0 & 0 & -I_{G_2} \\
0 & 0 & T_{12} & S_1 & 0 & -I_{G_1} \\
0 & 0 & 0 & S_2 & 0 & -I_{G_2} \\
0 & -T_{12} & 0 & T_{12} & I_{G_1} & 0 \\
-T_{21} & 0 & T_{21} & 0 & I_{G_2} & 0
\end{pmatrix} \tag{43}$$

The corresponding Schur complement system becomes

$$\Sigma_f \begin{pmatrix} \lambda_{G_1} \\ \lambda_{G_2} \end{pmatrix} = \chi_f$$

where

$$\Sigma_f = \begin{pmatrix}
I_{G_1} & (T_{12}S_1^{-1})^2 \\
T_{21}S_2^{-1}(I_{G_2} + T_{21}S_2^{-1})F_2
\end{pmatrix} \quad \text{and} \quad \chi_f = (T_{12}S_1^{-1}(I_{G_1} + T_{12}S_1^{-1})F_1) \tag{44}$$
Finally, the matrix associated to the optimality system for the functional $J_{tf}$ with mixed controls is:

$$
\begin{pmatrix}
S_{I_1 \Gamma_1} & 0 & 0 & S_{I_1 \Gamma_2} & 0 \\
0 & S_2 & 0 & 0 & -I_{T_2} \\
0 & -S_{I_1 \Gamma_1} R_{12} & S_{I_1 \Gamma_1} & 0 & 0 \\
T_{21} & 0 & 0 & S_2 & 0 \\
0 & -M_{I_1 \Gamma_1} R_{12} & M_{I_1 \Gamma_1} & 0 & 0 \\
\end{pmatrix}
$$

Its corresponding Schur complement system becomes

$$
\Sigma_{tf}(X_{I_1 \Gamma_1} X_{I_2 \Gamma_2}) = X_{tf}
$$

where

$$
\Sigma_{tf} = \begin{pmatrix} M_{I_1 \Gamma_1} (I_{T_2} - (R_{12} S_{I_1 \Gamma_1} S_{I_1 \Gamma_2})^2) \\
-I_{T_2} - (T_{21} S_{I_2}^2)^2 & \end{pmatrix}
$$

and

$$
X_{tf} = \begin{pmatrix} M_{I_1 \Gamma_1} R_{12} S_{I_1 \Gamma_1} - S_{I_1 \Gamma_1} R_{12} S_{I_1 \Gamma_2} \\
T_{21} S_{I_1 \Gamma_1} \end{pmatrix} F_1 (T_{21} + T_{21} S_{I_2}^2) F_2
$$

4. NUMERICAL RESULTS

4.1. Test cases with respect to an analytic solution

We consider the domain $\Omega = (0, 1) \times (0, 2)$ with $\Omega_1 = (0, 1) \times (1 - \delta/2, 2)$ and $\Omega_2 = (0, 1) \times (0, 1 + \delta/2)$, $\delta > 0$ being a suitable parameter characterizing the width of the overlapping region. The viscosity $\nu$ is set to 1, while the force $f$ and the boundary conditions are chosen such that the Stokes problem admits the solution $u = (exp(y), -exp(x))^T$ and $p = exp(x) \sin(y)$. Concerning the boundary conditions, we impose Neumann conditions on the boundary $1 \times (0, 2)$ while Dirichlet boundary conditions are imposed on the remaining boundaries. We compute the solution of the optimality system using the BiCGStab method on the Schur complement setting the tolerance to $10^{-9}$.

First, we consider the case of an overlap with fixed width $\delta = 0.2$. We use both Taylor-Hood elements with three computational meshes characterized by $h = 2^{-2}, 2^{-3}, 2^{-4}$, and stabilized $h_p$-FEM $Q_p - Q_p$. In the latter case, we consider 4 x 5 quad elements in each subdomain $\Omega$, 4 x 1 elements in $\Omega_{12}$, and each quad element has sides of length $h = 2^{-2}$.

In tables [I][II][III], we report the number of iterations required to converge, the computed infimum of the cost functional $J_{tf}$ and the errors

$$
e_n = \left(\left|u_1 - u_{1,h}\right|^2_{H^1(\Omega_1)} + \left|u_2 - u_{2,h}\right|^2_{H^1(\Omega_2)}\right)^{1/2},$$

where $u_{1,h}, u_{2,h}$ are the discrete solutions approximating the solutions of system $S_{tf}$.
Finally, we carry out a convergence test with Taylor-Hood elements setting $\delta = h$ and letting $h \to 0$. Also in this case we can see that the number of iterations required to converge grows when $h$ decreases. Results are reported in table IV.

<table>
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<th>$\delta$</th>
<th>#iter</th>
<th>$J_{tf}$</th>
<th>$e^p_{1,0}$</th>
<th>$e^p_{1,2,0}$</th>
<th>$e^h_{1,2,0}$</th>
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<td>3.922e-07</td>
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<td>15</td>
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<td>25</td>
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<td>5.088e-06</td>
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</tr>
<tr>
<td>0.02</td>
<td>71</td>
<td>3.369e-16</td>
<td>2.699e-05</td>
<td>4.974e-05</td>
<td>2.856e-09</td>
</tr>
<tr>
<td>0.01</td>
<td>250</td>
<td>4.208e-04</td>
<td>1.614e+01</td>
<td>2.909e+01</td>
<td>2.056e+03</td>
</tr>
</tbody>
</table>

These numerical results show that the ICDD method is not very effective especially when considering small overlapping regions. This behavior may due to the fact that the functional $J_t$ involves no information on the pressure fields in the overlap, since it imposes only the continuity of velocities on the interfaces.

The number of iterations is independent of the mesh size $h$ and of the polynomial degree $p$. However, a dependence on the size of the overlap can be estimated as

$$\text{#iter} \sim C\delta^{-1},$$

for a suitable positive constant $C > 0$.

We consider now the case of Neumann and mixed controls.

First, we consider the case of an overlap with fixed width $\delta = 0.2$. The setting and the discretization are the same used before. In tables V and VII we report the number of iterations and the computed errors for the case of the functional $J_f$ using Taylor-Hood and stabilized $Q_p - Q_p$ approximations, respectively, while in tables VIII and IX we report the results obtained for the functional $J_{tf}$.

Then, we consider the case where the width of the overlap tends to zero on a fixed computational mesh. Results are shown in tables X and XII for the Taylor-Hood elements with $h = 0.04$ and in tables XI and XIII for the stabilized $Q_p - Q_p$ elements with $p = 4$. Both functionals $J_f$ and $J_{tf}$ are used.

<table>
<thead>
<tr>
<th>$h$</th>
<th>#iter</th>
<th>$J_{tf}$</th>
<th>$e^p_{1,0}$</th>
<th>$e^h_{1,2,0}$</th>
<th>$e^p_{1,2,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^{-2}</td>
<td>6</td>
<td>1.211e-16</td>
<td>1.126e-02</td>
<td>8.778e-03</td>
<td>5.397e-05</td>
</tr>
<tr>
<td>2^{-3}</td>
<td>6</td>
<td>2.516e-16</td>
<td>2.829e-03</td>
<td>2.143e-03</td>
<td>4.352e-06</td>
</tr>
<tr>
<td>2^{-4}</td>
<td>6</td>
<td>1.445e-16</td>
<td>7.082e-04</td>
<td>5.314e-04</td>
<td>3.417e-07</td>
</tr>
</tbody>
</table>

Finally, we study the behavior of the ICDD method with functionals $J_f$ and $J_{tf}$ using Taylor-Hood elements setting $\delta = h$ and letting $h \to 0$. Results are reported in tables XIV and XV.

Differently from the case of Dirichlet controls with functional $J_t$, we can see that both functionals $J_f$ and $J_{tf}$ require a much lower number of iterations to converge. This shows that controlling the pressure and not only the velocity on the interfaces is crucial for the Stokes problem.

Moreover, we can see that the best convergence results are obtained with mixed controls and functional $J_{tf}$: as a matter of fact, in this case the number of iterations is independent from the mesh size $h$, from the degree $p$ of polynomial used, and from the measure $\delta$ of the overlap.

Neumann controls with functional $J_f$ also provide a number of iterations independent of the mesh size $h$ and of the polynomial degree $p$. However, a dependence on the size of the overlap can be noticed as

$$\text{#iter} \sim C\delta^{-1/2},$$

for a suitable positive constant $C > 0$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>#iter</th>
<th>$J_{tf}$</th>
<th>$e^p_{1,0}$</th>
<th>$e^h_{1,2,0}$</th>
<th>$e^p_{1,2,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^{-2}</td>
<td>6</td>
<td>1.211e-16</td>
<td>1.126e-02</td>
<td>8.778e-03</td>
<td>5.397e-05</td>
</tr>
<tr>
<td>2^{-3}</td>
<td>6</td>
<td>2.516e-16</td>
<td>2.829e-03</td>
<td>2.143e-03</td>
<td>4.352e-06</td>
</tr>
<tr>
<td>2^{-4}</td>
<td>6</td>
<td>1.445e-16</td>
<td>7.082e-04</td>
<td>5.314e-04</td>
<td>3.417e-07</td>
</tr>
</tbody>
</table>

We summarize the convergence results of the functional $J_f$ with Taylor-Hood elements with degree $p = 4$ and $p = 6$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>#iter</th>
<th>$J_{tf}$</th>
<th>$e^p_{1,0}$</th>
<th>$e^h_{1,2,0}$</th>
<th>$e^p_{1,2,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^{-2}</td>
<td>6</td>
<td>1.211e-16</td>
<td>1.126e-02</td>
<td>8.778e-03</td>
<td>5.397e-05</td>
</tr>
<tr>
<td>2^{-3}</td>
<td>6</td>
<td>2.516e-16</td>
<td>2.829e-03</td>
<td>2.143e-03</td>
<td>4.352e-06</td>
</tr>
<tr>
<td>2^{-4}</td>
<td>6</td>
<td>1.445e-16</td>
<td>7.082e-04</td>
<td>5.314e-04</td>
<td>3.417e-07</td>
</tr>
</tbody>
</table>
TABLE IX: Test case with analytic solution. Results for the functional $J_{f}$ with stabilized $Q_{p} - Q_{p}$ elements with respect to different polynomial degrees $p$. Fixed overlap with $\delta = 0.2$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$#$iter $J_{f}$</th>
<th>$e_{1}^{1}$</th>
<th>$e_{1}^{2}$</th>
<th>$e_{2}^{1}$</th>
<th>$e_{2}^{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
<td>2.043e-16</td>
<td>0.000e+00</td>
<td>0.000e+00</td>
<td>0.000e+00</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>3.044e-16</td>
<td>0.000e+00</td>
<td>0.000e+00</td>
<td>0.000e+00</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>2.334e-16</td>
<td>0.000e+00</td>
<td>0.000e+00</td>
<td>0.000e+00</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>2.185e-16</td>
<td>0.000e+00</td>
<td>0.000e+00</td>
<td>0.000e+00</td>
</tr>
</tbody>
</table>

4.2. A test case without analytic solution

We consider the computational domain $\Omega = (0, 1) \times (0, 2)$ with $\Omega_{1} = (0, 1) \times (1 - \delta/2, 2)$ and $\Omega_{2} = (0, 1) \times (0, 1 + \delta/2)$, as represented schematically in Figure 2. The force is set to $f = 0$ and the viscosity is $\nu = 2, e^{-3}$. We impose homogeneous Neumann boundary conditions on the edges $l_{4}$ and $l_{7}$. On the remaining boundaries, apart from the edge $l_{0}$, we impose homogeneous Dirichlet boundary conditions unless on $\{0\} \times (1, 2)$ where we set a parabolic profile with maximum equal to 1.

On the edge $l_{0}$, we may impose either homogeneous Neumann or Dirichlet boundary conditions to compare the behavior of the different methods that we have studied. In particular, we want to show that the functional $J_{f}$ with Dirichlet controls will not provide a correct solution when $l_{0}$ is set as a Dirichlet boundary, since this case violates the Assumption [2].

For this problem, besides the errors $e_{12}^{u}$ and $e_{12}^{p}$ on the overlap, we also compute

TABLE XI: Test case with analytic solution. Results for the functional $J_{f}$ with Taylor-Hood elements with $h = 0.04$ and $\delta \rightarrow 0$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$#$iter $J_{f}$</th>
<th>$e_{1}^{1}$</th>
<th>$e_{1}^{2}$</th>
<th>$e_{2}^{1}$</th>
<th>$e_{2}^{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>9</td>
<td>2.438e-18</td>
<td>3.889e-07</td>
<td>3.035e-07</td>
<td>3.230e-09</td>
</tr>
<tr>
<td>0.1</td>
<td>15</td>
<td>3.759e-17</td>
<td>2.465e-07</td>
<td>2.579e-09</td>
<td>2.059e-09</td>
</tr>
<tr>
<td>0.05</td>
<td>25</td>
<td>4.783e-15</td>
<td>6.835e-07</td>
<td>7.114e-07</td>
<td>5.991e-08</td>
</tr>
<tr>
<td>0.02</td>
<td>96</td>
<td>2.052e-16</td>
<td>6.653e-07</td>
<td>8.673e-07</td>
<td>3.315e-08</td>
</tr>
<tr>
<td>0.01</td>
<td>250*</td>
<td>5.751e-04</td>
<td>8.371e-01</td>
<td>9.842e-01</td>
<td>5.206e-02</td>
</tr>
</tbody>
</table>

TABLE XII: Test case with analytic solution. Results for the functional $J_{f}$ with Taylor-Hood elements with $h = 0.04$ and $\delta \rightarrow 0$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$#$iter $J_{f}$</th>
<th>$e_{1}^{1}$</th>
<th>$e_{1}^{2}$</th>
<th>$e_{2}^{1}$</th>
<th>$e_{2}^{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>6</td>
<td>1.861e-16</td>
<td>3.890e-07</td>
<td>3.035e-07</td>
<td>3.230e-09</td>
</tr>
<tr>
<td>0.1</td>
<td>7</td>
<td>7.935e-16</td>
<td>4.621e-07</td>
<td>4.239e-07</td>
<td>8.769e-10</td>
</tr>
<tr>
<td>0.05</td>
<td>9</td>
<td>6.422e-16</td>
<td>5.903e-07</td>
<td>4.877e-07</td>
<td>4.034e-10</td>
</tr>
<tr>
<td>0.02</td>
<td>11</td>
<td>1.103e-14</td>
<td>5.421e-07</td>
<td>5.183e-07</td>
<td>9.533e-10</td>
</tr>
<tr>
<td>0.01</td>
<td>8</td>
<td>4.678e-14</td>
<td>5.511e-07</td>
<td>5.511e-10</td>
<td>4.550e-08</td>
</tr>
</tbody>
</table>

TABLE XIII: Test case with analytic solution. Results for the functional $J_{f}$ with stabilized $Q_{p} - Q_{p}$ elements with respect to different polynomial degrees $p$ for $\delta \rightarrow 0$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$#$iter $J_{f}$</th>
<th>$e_{1}^{1}$</th>
<th>$e_{1}^{2}$</th>
<th>$e_{2}^{1}$</th>
<th>$e_{2}^{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>6</td>
<td>2.334e-16</td>
<td>3.890e-07</td>
<td>3.035e-07</td>
<td>3.230e-09</td>
</tr>
<tr>
<td>0.1</td>
<td>7</td>
<td>7.935e-16</td>
<td>4.621e-07</td>
<td>4.239e-07</td>
<td>8.769e-10</td>
</tr>
<tr>
<td>0.05</td>
<td>9</td>
<td>6.422e-16</td>
<td>5.903e-07</td>
<td>4.877e-07</td>
<td>4.034e-10</td>
</tr>
<tr>
<td>0.02</td>
<td>11</td>
<td>1.103e-14</td>
<td>5.421e-07</td>
<td>5.183e-07</td>
<td>9.533e-10</td>
</tr>
<tr>
<td>0.01</td>
<td>8</td>
<td>4.678e-14</td>
<td>5.511e-07</td>
<td>5.511e-10</td>
<td>4.550e-08</td>
</tr>
</tbody>
</table>

TABLE XIV: Test case with analytic solution. Results for the functional $J_{f}$ with Taylor-Hood elements with $h = 0.04$ and $\delta \rightarrow 0$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$#$iter $J_{f}$</th>
<th>$e_{1}^{1}$</th>
<th>$e_{1}^{2}$</th>
<th>$e_{2}^{1}$</th>
<th>$e_{2}^{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>6</td>
<td>1.861e-16</td>
<td>3.890e-07</td>
<td>3.035e-07</td>
<td>3.230e-09</td>
</tr>
<tr>
<td>0.1</td>
<td>7</td>
<td>7.935e-16</td>
<td>4.621e-07</td>
<td>4.239e-07</td>
<td>8.769e-10</td>
</tr>
<tr>
<td>0.05</td>
<td>9</td>
<td>6.422e-16</td>
<td>5.903e-07</td>
<td>4.877e-07</td>
<td>4.034e-10</td>
</tr>
<tr>
<td>0.02</td>
<td>11</td>
<td>1.103e-14</td>
<td>5.421e-07</td>
<td>5.183e-07</td>
<td>9.533e-10</td>
</tr>
<tr>
<td>0.01</td>
<td>8</td>
<td>4.678e-14</td>
<td>5.511e-07</td>
<td>5.511e-10</td>
<td>4.550e-08</td>
</tr>
</tbody>
</table>
TABLE XV: Test case with analytic solution. Results for the functional $J_{if}$ with Taylor-Hood elements with $\delta = h$ and $\delta \to 0$.

<table>
<thead>
<tr>
<th>$\delta = h$</th>
<th>#iter $\inf{J_{if}}$</th>
<th>$e_{\tau}^f$</th>
<th>$e_{\tau}^p$</th>
<th>$e_{\tau 2,0}$</th>
<th>$e_{\tau 2,12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/3$</td>
<td>5 4.337e-18 2.487e-02 2.051e-02 7.757e-04 8.554e-03</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1/6$</td>
<td>6 1.596e-15 6.709e-03 5.411e-03 4.145e-05 8.970e-04</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2/25$</td>
<td>7 7.951e-15 1.572e-03 1.275e-03 2.503e-06 1.067e-04</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1/25$</td>
<td>7 5.075e-15 3.907e-04 3.141e-04 1.596e-07 1.332e-05</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

FIG. 2: Schematic representation of the computational domain.

TABLE XVI: Test case without analytic solution. Dirichlet boundary condition on $l_6$. Results for the functionals $J_1$ (top), $J_{if}$ (mid) and $J_{it}$ (bottom) with Taylor-Hood elements with fixed $h = 0.04$ and $\delta \to 0$.

<table>
<thead>
<tr>
<th>$\delta = h$</th>
<th>#iter $\inf{J_1}$</th>
<th>$e_{\tau}^f$</th>
<th>$e_{\tau}^p$</th>
<th>$e_{\tau 2,0}$</th>
<th>$e_{\tau 2,12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4h$</td>
<td>9 5.197e-21 2.805e-01 3.740e-03 9.122e-04 4.568e-03</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3h$</td>
<td>12 5.289e-23 2.967e-01 3.917e-03 1.002e-03 4.261e-03</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h$</td>
<td>31 8.042e-22 1.921e-01 2.231e-04 9.904e-05 1.510e-03</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Finally, let us consider a test case in which the interface and their behaviors are similar to those observed in the previous tests with analytic solution. The functional $J_{if}$ associated to mixed controls is the one that converges in the lowest number of iterations with a slight dependence on $\delta$. Results are reported in table XVII for Taylor-Hood elements and in table XIX for stabilized $Q_p = Q_p$ elements with $p = 6$.

In Figure 6 we show the single-domain solution, while in Figures 7 and 8 we show, respectively, the solutions obtained through minimization of the functional $J_1$ and $J_{if}$. We can see that, although the functional $J_1$ has no control on the pressure, the Neumann boundary condition on the edge $l_6$ allows the pressure to match almost perfectly in the overlapping region. Notice that the difference shown in Figure 7 is of the same order of the errors reported in tables XVIII and XIX.

Finally, let us consider a test case in which the interface is a piecewise linear curve (identified by element edges), as shown in Fig. 9. We compute the solution by imposing a Neumann boundary condition on the boundary $l_6$ considering stabilized $Q_p = Q_p$ elements with $p = 6$ and $\delta \to 0$. The iterations numbers shown in table XXI behave similarly to those presented in the third block of table XIX the algorithm is not strongly influenced by the shape of the interface.

To assess the robustness of the method with respect to the viscosity coefficient $\nu$, we compute the solution of the problem with Neumann boundary condition on $l_6$ using the ICDD method associated to the functional $J_{if}$, the one that provided the best results in the previous tests. We consider a discretization by Taylor-Hood elements on a mesh with fixed $h = 0.04$ and $\delta \to 0$ and we set the viscosity $\nu = 10^{-2}, 10^{-4}, 10^{-6}$. Numerical results are reported in tables XXII and XXIII clearly they show that the method is robust with respect to variations of the parameter $\nu$.

5. ANALYSIS OF THE ICDD METHOD FOR THE STOKES PROBLEM

In this section we analyze the ICDD method that we have presented in the previous sections with the aim of guaranteeing the well-posedness of the minimization problem. We begin with the analysis in the continuous case with Dirichlet controls.

5.1. Analysis of the optimal control problem with Dirichlet controls

For $i = 1, 2$, we introduce the following spaces:

$$\Lambda_i = \{ \mu \in [H^{1/2}(\Gamma_i)]^d : \exists v \in [H^{1}(\Omega_i)]^d, \mu = v \text{ on } \Gamma_i \text{ and } v = 0 \text{ on } \Gamma^R \} \quad (45)$$

$$\Lambda_{i,0} = \{ \mu \in \Lambda_i : \int_{\Gamma_i} \mu \cdot n = 0 \} \quad (46)$$
We will denote by

$$\Lambda_i^D = \begin{cases} \Lambda_i & \text{if } \partial \Omega_i \cap \Gamma_N \neq \emptyset \\ \Lambda_{i,0} & \text{if } \partial \Omega_i \cap \Gamma_N = \emptyset, \end{cases} \quad i = 1, 2, \quad (47)$$

the spaces of admissible Dirichlet controls. Moreover, we will denote

$$\Lambda^D = \Lambda_1^D \times \Lambda_2^D. \quad (48)$$

For $i = 1, 2$, we consider two unknown control functions $\lambda_i \in \Lambda_i^D$ and the associated state problems

$$\begin{align*}
- \nabla \cdot T(u_i^{\lambda_i,f}, p_i^{\lambda_i,f}) &= f & \text{in } \Omega_i \\
\nabla \cdot u_i^{\lambda_i,f} &= 0 & \text{in } \Omega_i \\
u_i^{\lambda_i,f} &= \lambda_i & \text{on } \Gamma_i
\end{align*} \quad (49)$$

with suitable homogeneous boundary conditions on $\partial \Omega_i \setminus \Gamma_i$. If $\Gamma_{1,2} = \emptyset$, we add the constraint $\int_{\Gamma_1} p_i^{\lambda_i,f} = 0$. The unknown controls on the interface are obtained by solving the minimization problem

$$\inf_{\Delta = (\lambda_1, \lambda_2) \in \Lambda^D} \left[ J_f(\Delta) := \frac{1}{2} \| u_1^{\lambda_1,f} - u_2^{\lambda_2,f} \|^2_{L^2(\Gamma_1 \cup \Gamma_2)} \right] \quad (50)$$

where, for the sake of simplicity, we adopt the same notation as in the discrete case.

This yields an optimal control problem where both the control functions and the observations are of boundary (interface) type.

\begin{table}[ht]
\centering
\caption{Test case without analytic solution. Dirichlet boundary condition on $\Gamma$. Results for the functionals $J(f)$ (top), $J_2$ (mid) and $J_1$ (bottom) with stabilized $\mathbb{Q}_p - \mathbb{Q}_2$ elements with fixed $p = 6$ and $\delta \to 0$.}
\begin{tabular}{c|c|c|c|c|c|c}
\hline
$\delta$ & #iter & $J(f)$ & $e_1^{\lambda}$ & $e_2^{\lambda}$ & $e_{12,0}$ & $e_{12,2,0}$ \\
\hline
0.2 & 8 & 1.582e-20 & 3.151e-02 & 1.239e-04 & 2.158e-05 & 1.176e-04 \\
0.05 & 25 & 1.698e-22 & 4.565e-01 & 1.389e-03 & 2.740e-05 & 8.364e-04 \\
0.02 & 65 & 4.433e-22 & 2.349e+00 & 7.292e-03 & 4.594e-05 & 2.861e-03 \\
0.01 & 214 & 2.483e-23 & 6.278e+00 & 2.021e-02 & 4.281e-05 & 5.798e-03 \\
\hline
\end{tabular}
\end{table}

\begin{table}[ht]
\centering
\caption{Test case without analytic solution. Dirichlet boundary condition on $\Gamma$. Results for the functionals $J_2$ (top), $J_{12}$ (mid) and $J_2$ (bottom) with stabilized $\mathbb{Q}_p - \mathbb{Q}_2$ elements with fixed $p = 6$ and $\delta \to 0$.}
\begin{tabular}{c|c|c|c|c|c|c}
\hline
$\delta$ & #iter & $J_1$ & $e_1^{\lambda}$ & $e_2^{\lambda}$ & $e_{12,0}$ & $e_{12,2,0}$ \\
\hline
0.2 & 6 & 1.955e-26 & 3.475e-03 & 3.354e-06 & 1.676e-04 & 1.605e-05 \\
0.1 & 11 & 1.431e-25 & 4.699e-03 & 1.056e-05 & 4.011e-04 & 1.344e-05 \\
0.05 & 22 & 1.154e-24 & 1.235e-02 & 3.908e-05 & 1.148e-03 & 2.123e-05 \\
0.02 & 55 & 1.241e-24 & 1.037e-02 & 2.953e-05 & 6.474e-04 & 1.436e-05 \\
0.01 & 165 & 2.021e-23 & 6.362e-02 & 1.025e-02 & 1.660e-03 & 2.036e-05 \\
\hline
\end{tabular}
\end{table}

FIG. 3: Test case without analytic solution. Dirichlet boundary condition on $\Gamma$. Reference monodomain solution computed using Taylor-Hood finite elements.

Thanks to the linearity of the problem, we have $u_i^{\lambda_i,f} = u_i^{\lambda_i,0} + u_i^{0,f}$ and $p_i^{\lambda_i,f} = p_i^{\lambda_i,0} + p_i^{0,f}$. For the sake of simplicity, we will indicate $u_i^{\lambda_i} = u_i^{\lambda_i,0}$ and $p_i^{\lambda_i} = p_i^{\lambda_i,0}$ and $\Delta = (u_1^{\lambda_1}, u_2^{\lambda_2}), p^\Delta = (p_1^{\lambda_1}, p_2^{\lambda_2})$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{Test case without analytic solution. Dirichlet boundary condition on $\Gamma$. Reference monodomain solution computed using Taylor-Hood finite elements.}
\end{figure}
Then, we can equivalently express the cost functional as

\[
J_t(\lambda) = \frac{1}{2} \| u_1^\lambda - u_2^\lambda \|_{L^2(\Gamma_1 \cup \Gamma_2)}^2 + (u_1^\lambda, u_2 \lambda^\tau, u_1^\lambda, u_2 \lambda^\tau, u_2 \lambda^\tau, f)_{L^2(\Gamma_1 \cup \Gamma_2)} + \frac{1}{2} \| u_1^\lambda - u_2^\lambda \|_{L^2(\Gamma_1 \cup \Gamma_2)}^2
\]

(51)

In this section we will denote \( \| \Delta \|_D = \| u_1^\lambda - u_2^\lambda \|_{L^2(\Gamma_1 \cup \Gamma_2)} \).

**Lemma 5.1** If the boundary conditions imposed on the Stokes problem (1) satisfy Assumption 2.7, then \( \| \Delta \|_D \) defines a norm on the space \( \Lambda^D \).
TABLE XVIII: Test case without analytic solution. Neumann boundary condition on \( h \). Results for the functionals \( J_f \) (top), \( J_f \) (mid) and \( J_f \) (bottom) with Taylor-Hood elements with fixed \( h = 0.04 \) and \( \delta \to 0 \).

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>#iter</th>
<th>inf ( J_f )</th>
<th>( e_1^p )</th>
<th>( e_2^p )</th>
<th>( e_{120}^p )</th>
<th>( e_{12,0}^p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5h</td>
<td>8</td>
<td>3.771e-21</td>
<td>1.164e-02</td>
<td>2.069e-04</td>
<td>7.870e-04</td>
<td>2.954e-04</td>
</tr>
<tr>
<td>4h</td>
<td>9</td>
<td>1.942e-20</td>
<td>2.075e-02</td>
<td>3.106e-04</td>
<td>8.418e-04</td>
<td>4.235e-04</td>
</tr>
<tr>
<td>2h</td>
<td>17</td>
<td>1.470e-25</td>
<td>2.984e-03</td>
<td>2.994e-05</td>
<td>7.864e-05</td>
<td>3.913e-05</td>
</tr>
<tr>
<td>h</td>
<td>34</td>
<td>1.077e-24</td>
<td>5.109e-02</td>
<td>5.826e-04</td>
<td>2.688e-04</td>
<td>4.050e-04</td>
</tr>
</tbody>
</table>

\[
\delta \to 0 \Rightarrow \text{results converge within 250 iterations.}
\]

Proof. Since \( \| \Delta \|_D \) is obviously a semi-norm on \( \Lambda^D \), we only have to prove that, \( \| \Delta \|_D = 0 \), then \( \Lambda = 0 \). Obviously, \( \| \Delta \|_D = 0 \) implies that \( u^\Lambda = u^{22} \) a.e. on \( \Gamma_1 \cup \Gamma_2 \). As \( u^\Lambda, p^\Lambda \) is the solution of (49) with \( f = 0 \), \( (w, q) = (u^\Lambda u_{12} - u^\Lambda u_{21}, \eta^\Lambda - p^\Lambda_{121}) \) satisfies

\[
-\nabla \cdot T(w, q) = 0 \quad \text{in} \quad \Omega_{12} \\
\nabla w = 0 \quad \text{in} \quad \Omega_{12} \\
w = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2
\]

with suitable homogeneous boundary conditions on \( \partial \Omega_{12} \). Since \( u^\Lambda \) belongs to \( H^1(\Omega_2) \), condition (52) has to be interpreted in the sense of traces of zeroth order of \( H^1 \) functions on \( \Gamma_1 \cup \Gamma_2 \).

Following the same arguments used in the proof of Proposition 2.1 it can be shown that problem (52) is well-posed and its solution is \( w = 0 \) and \( q = \text{const} \). Thus, \( u^\Lambda u_{12} = u^\Lambda_{22} \) and \( p^\Lambda_{12} + C_1 = p^\Lambda_{22} + C_2 \) a.e. in \( \Omega_{12} \) with \( C_1, C_2 \in \mathbb{R}, \eta = C_2 - C_1 \), and we can define

\[
\mathfrak{W} = \begin{cases} \\
\begin{array}{l}
u^\Lambda_1 \quad \text{in} \quad \Omega_1 \setminus \Omega_{12} \\
u^\Lambda_1 u^\Lambda_{22} \quad \text{in} \quad \Omega_{12} \\
u^\Lambda_1 \quad \text{in} \quad \Omega_2 \setminus \Omega_{12} \end{array} \\
\end{cases}
\]

and

\[
\mathfrak{P} = \begin{cases} \\
\begin{array}{l}
u^\Lambda_1 + C_1 \quad \text{in} \quad \Omega_1 \setminus \Omega_{12} \\
u^\Lambda_1 + C_1 = p^\Lambda_{22} + C_2 \quad \text{in} \quad \Omega_{12} \\

\end{array} \\
\end{cases}
\]

TABLE XIX: Test case without analytic solution. Neumann boundary condition on \( h \). Results for the functionals \( J_f \) (top), \( J_f \) (mid) and \( J_f \) (bottom) with stabilized \( \mathbb{Q}_p - \mathbb{Q}_p \) elements with fixed \( p = 6 \) and \( \delta \to 0 \). By \( \text{we denote that the method did not converge within 250 iterations.} \)

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>#iter</th>
<th>inf ( J_f )</th>
<th>( e_1^p )</th>
<th>( e_2^p )</th>
<th>( e_{120}^p )</th>
<th>( e_{12,0}^p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>8</td>
<td>7.054e-24</td>
<td>1.398e-02</td>
<td>8.173e-05</td>
<td>1.539e-04</td>
<td>6.044e-05</td>
</tr>
<tr>
<td>0.1</td>
<td>14</td>
<td>1.005e-25</td>
<td>3.046e-02</td>
<td>1.121e-04</td>
<td>1.233e-04</td>
<td>8.945e-05</td>
</tr>
<tr>
<td>0.05</td>
<td>25</td>
<td>3.182e-23</td>
<td>3.101e-02</td>
<td>1.964e-04</td>
<td>8.335e-05</td>
<td>1.249e-04</td>
</tr>
<tr>
<td>0.02</td>
<td>65</td>
<td>3.579e-23</td>
<td>2.969e-02</td>
<td>3.047e-04</td>
<td>6.732e-05</td>
<td>1.272e-03</td>
</tr>
<tr>
<td>0.01</td>
<td>211</td>
<td>8.323e-21</td>
<td>6.644e-02</td>
<td>1.995e-05</td>
<td>8.936e-05</td>
<td>5.879e-03</td>
</tr>
</tbody>
</table>

TABLE XX: Test case without analytic solution. Piecewise linear interfaces. Neumann boundary condition on \( h \). Results for the functionals \( J_f \) with stabilized \( \mathbb{Q}_p - \mathbb{Q}_p \) elements with fixed \( p = 6 \) and \( \delta \to 0 \).

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>#iter</th>
<th>inf ( J_f )</th>
<th>( e_1^p )</th>
<th>( e_2^p )</th>
<th>( e_{120}^p )</th>
<th>( e_{12,0}^p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>5</td>
<td>1.208e-22</td>
<td>2.128e-02</td>
<td>7.894e-05</td>
<td>3.015e-04</td>
<td>5.537e-05</td>
</tr>
<tr>
<td>0.1</td>
<td>6</td>
<td>2.256e-21</td>
<td>1.660e-02</td>
<td>7.874e-05</td>
<td>2.277e-04</td>
<td>5.645e-05</td>
</tr>
<tr>
<td>0.05</td>
<td>7</td>
<td>2.385e-22</td>
<td>1.690e-02</td>
<td>7.153e-05</td>
<td>1.438e-04</td>
<td>5.033e-05</td>
</tr>
<tr>
<td>0.02</td>
<td>7</td>
<td>1.144e-20</td>
<td>1.373e-02</td>
<td>4.956e-05</td>
<td>6.421e-05</td>
<td>3.249e-05</td>
</tr>
<tr>
<td>0.01</td>
<td>7</td>
<td>6.640e-21</td>
<td>1.084e-02</td>
<td>3.657e-05</td>
<td>2.670e-05</td>
<td>2.421e-05</td>
</tr>
</tbody>
</table>

By construction, the pair \((\mathfrak{W}, \mathfrak{P})\) satisfies a Stokes problem in \( \Omega \) with null force and homogeneous boundary conditions with \( \Gamma_N \neq \emptyset \). This problem is well-posed and, in particular, \( \mathfrak{P} = 0 \) a.e. in \( \Omega \). This implies that \( \mathfrak{P} = 0 \) on \( \Gamma_1 \cup \Gamma_2 \) and, for \( i = 1, 2, \mathfrak{W}_i = 0 \) in \( \Lambda_i \).
\( \Lambda^D_h \subset \Lambda^D \subset \hat{\Lambda}^D \) and, at the discrete level, all norms are equivalent. Thus, this would not be a problem for the application that we have in mind. For the sake of notation, in the following we will still denote the completion of \( \Lambda^D \) by the same symbol.

**Theorem 5.1** Consider the minimization problem

\[
\inf_{\Delta^D \Lambda^D} J_t(\Delta^D), \tag{55}
\]

If Assumption 2.1 holds, problem (55) has a unique solution.
FIG. 8: Test case without analytic solution. Neumann boundary condition on \( \Omega_6 \). Solution computed by minimizing the functional \( J_t \) using Taylor-Hood finite elements.

FIG. 9: Computational mesh for stabilized \( Q_p - Q_p \) elements in the case of piecewise linear interfaces. In the figure \( \delta = 0.01 \).

Proof. For any \( \Delta \in \Lambda^D \), let us define
\[
\pi(\Delta, \mu) = \frac{1}{2} (u_1^{\lambda_1} - u_2^{\lambda_2}, u_1^{\mu_1} - u_2^{\mu_2})_{L^2(\Gamma_1 \cup \Gamma_2)},
\]
\[
L(\mu) = -\frac{1}{2} (u_1^{0, f} - u_2^{0, f}, u_1^{\mu_1} - u_2^{\mu_2})_{L^2(\Gamma_1 \cup \Gamma_2)}
\]
so that
\[
J_t(\lambda) = \pi(\lambda, \lambda) - 2L(\lambda) + \frac{1}{2} \|u_1^{0, f} - u_2^{0, f}\|_{L^2(\Gamma_1 \cup \Gamma_2)}^2.
\]

The bilinear form \( \pi : \Lambda^D \times \Lambda^D \to \mathbb{R} \) is symmetric by definition and, thanks to Lemma 5.1, is continuous and coercive with respect to the norm \( \|\cdot\|_D \). Moreover, \( L : \Lambda^D \to \mathbb{R} \) is a linear continuous functional. Then, being \( \Lambda^D, \|\cdot\|_D \) a Hilbert space (recall that now \( \Lambda^D \) denotes its completion with respect to the norm \( \|\cdot\|_D \)), applying classical results of calculus of variations (see, e.g., [12, Theorem 1.1]), the existence and uniqueness of the solution is guaranteed.

The Euler-Lagrange equation (56) follows by observing that, for all \( \lambda, \mu \in \Lambda^D \),
\[
(\Lambda^D)' \langle J_t'(\Delta), \mu \rangle_{\Lambda^D} = (u_1^{\lambda_1}, f - u_2^{\lambda_2}, u_1^{\mu_1} - u_2^{\mu_2})_{L^2(\Gamma_1 \cup \Gamma_2)} = 0
\]
for all \( \mu \in \Lambda^D \).

Remark 5.1 Notice that, although the definition of the functional \( J_t \) involves the difference between the traces of the velocity on \( \Gamma_1 \cup \Gamma_2 \) only, the requirement that \( \partial \Omega_{12} \cap \Gamma_N \neq \emptyset \) guarantees that the local pressures \( p_1 \) and \( p_2 \) will match in the overlapping region, i.e., \( p_1 = p_2 \) a.e. in \( \Omega_{12} \).
TABLE XXI: Test case without analytic solution. Neumann boundary condition on $\Omega_0$. Results obtained for the functional $J_{f_H}$ with Taylor-Hood elements with fixed $h = 0.04$ and $\delta \to 0$. The viscosity is $\nu = 10^{-2} \text{top}$, $\nu = 10^{-4} \text{mid}$, $\nu = 10^{-6} \text{bottom}$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\nu = 10^{-2}$</th>
<th>$\nu = 10^{-4}$</th>
<th>$\nu = 10^{-6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5h$</td>
<td>$2.688e-18$</td>
<td>$6.686e-03$</td>
<td>$3.452e-04$</td>
</tr>
<tr>
<td>$3h$</td>
<td>$6.553e-20$</td>
<td>$1.726e-03$</td>
<td>$2.858e-04$</td>
</tr>
<tr>
<td>$2h$</td>
<td>$8.063e-20$</td>
<td>$4.077e-04$</td>
<td>$6.166e-05$</td>
</tr>
<tr>
<td>$h$</td>
<td>$5.523e-19$</td>
<td>$1.150e-03$</td>
<td>$1.387e-04$</td>
</tr>
</tbody>
</table>

5.1.1. The optimality system for Dirichlet controls

After Theorem 5.1, we assume that Assumption 2.1 is satisfied. More in particular, we consider the case $\delta S_{12} \cap \Gamma_N \neq \emptyset$ and $\delta S_D \neq \emptyset$ so that the constants $C_1$ and $C_2$ of Lemma 5.1 are both null. In the other cases, we would require that $p_i, q_i \in Q^i_0$, and the non-null constants $C_1, C_2$ are those identified in the proof of Proposition 2.1.

The Euler-Lagrange equation (60) becomes:

$$
\langle \Lambda^D \rangle \langle J_{f_H}(\Lambda), \mu \rangle_{\Lambda^D} = \int_{\Gamma_1 \cup \Gamma_2} (u_1^{\lambda_1} - f - u_2^{\lambda_2}) (u_1^{\mu_1} - u_2^{\mu_2}) = 0
$$

for all $\mu \in \Lambda^D$.

Solving equation (57) is equivalent to solving the following optimality system: find $\Lambda = (\lambda_1, \lambda_2) \in \Lambda^D$ and, for $i = 1, 2, (u_i, p_i) \in V_{i, 0} \times Q_{i, 0}$, $(w_i, q_i) \in V_{i, 0} \times Q_{i, 0}$ such that

$$
\begin{align*}
-\text{div} \mathbf{T}(u_i, p_i) &= f \quad \text{in } \Omega_i \\
\text{div } u_i &= 0 \quad \text{in } \Omega_i \\
u_i &= \lambda_i \quad \text{on } \Gamma_i \\
\mathbf{T}(u_i, p_i) \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_N
\end{align*}
$$

(58)

TABLE XXII: Test case without analytic solution. Neumann boundary condition on $\Omega_0$. Results obtained for the functional $J_{f_H}$ with stabilized $Q_p - Q_0$ elements with fixed $p = 0$ and $\delta \to 0$. The viscosity is $\nu = 10^{-2} \text{top}$, $\nu = 10^{-4} \text{mid}$, $\nu = 10^{-6} \text{bottom}$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\nu = 10^{-2}$</th>
<th>$\nu = 10^{-4}$</th>
<th>$\nu = 10^{-6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4h$</td>
<td>$5.438e-20$</td>
<td>$1.660e-02$</td>
<td>$3.937e-04$</td>
</tr>
<tr>
<td>$3h$</td>
<td>$5.672e-21$</td>
<td>$1.690e-02$</td>
<td>$3.576e-04$</td>
</tr>
<tr>
<td>$2h$</td>
<td>$2.684e-19$</td>
<td>$1.373e-02$</td>
<td>$2.478e-04$</td>
</tr>
<tr>
<td>$h$</td>
<td>$1.594e-19$</td>
<td>$1.048e-02$</td>
<td>$1.829e-04$</td>
</tr>
</tbody>
</table>

5.1. The optimality system for Neumann controls

Proof. Let $\Lambda$ be the solution of (59). Theorem 5.1 guarantees that such solution exists and is unique. Then, it is also a solution of (58) (60). Indeed, the solution satisfies (57) which implies that $u_1^{\lambda_1} = u_2^{\lambda_2}$ on $\Gamma_1 \cup \Gamma_2$. As a consequence the solutions $(w_i, q_i)$ of (59) are identically null and (60) is satisfied.

We prove now that this solution is unique. Consider first
the case \( f = 0 \). We define the operator \( \chi : \Lambda^D \rightarrow (\Lambda^D)' \),
\[
(\Lambda^D)'(\chi(\Lambda), \mu')_{\Lambda^D} = \int_{\Gamma_1} ((u_1^{\lambda_1} - u_2^{\lambda_2}) + w_2^\Delta) \mu_1 \, d\Gamma + \int_{\Gamma_2} (u_1^{\lambda_1} - u_2^{\lambda_2}) + w_2^\Delta \mu_2 \, d\Gamma,
\]
where, for \( i = 1, 2 \), \( u_i^{\lambda_i} \), \( w_i^\Delta \) are solutions of (58) and (59), respectively, with \( \Gamma = 0 \). The operator \( \chi \) is linear and continuous, and \( ker(\chi) = \{0\} \). Indeed, thanks to (59), \( w_i^\Delta \in [H^{1/2}(\Gamma_i)]^d \) and, if \( \Lambda \in ker(\chi) \), due to (61),
\[
w_2^\Delta = -(u_1^{\lambda_1} - u_2^{\lambda_2}) \text{ on } \Gamma_1 \text{ and } w_2^\Delta = (u_1^{\lambda_1} - u_2^{\lambda_2}) \text{ on } \Gamma_2.
\]
Thus, for \( i = 1, 2 \), \( j = 3 - i \), \( w_i^\Delta \) satisfies the system
\[
-\text{div} \, T(w_i^\Delta, q_i^\Delta) = 0 \quad \text{in } \Omega_i
\]
\[
d\text{div} \, w_i^\Delta = 0 \quad \text{in } \Omega_i
\]
\[
\tilde{w}_i^\Delta = -w_i^\Delta \quad \text{on } \Gamma_i
\]
\[
w_i^\Delta = 0 \quad \text{on } \Gamma_i^0
\]
\[
T(w_i^\Delta, q_i^\Delta) \cdot n = 0 \quad \text{on } \Gamma_N^i
\]
We define \( \tilde{w} = w_1^\Delta|_{\Omega_1} + w_2^\Delta|_{\Omega_2} \) and \( \tilde{q} = q_1|_{\Omega_1} + q_2|_{\Omega_2} \) in \( \Omega_{12} \).

The system is identically null. Thus, \( w_i^\Delta = -w_i^\Delta \) and \( q_i^\Delta = -q_i^\Delta \) in \( \Omega_{12} \) and we can define
\[
w = \begin{cases} 
w_1^\Delta & \text{in } \Omega_1 \setminus \Omega_{12} \\
w_2^\Delta & \text{in } \Omega_2 \setminus \Omega_{12} \end{cases}
\]
and
\[
q = \begin{cases} q_1^\Delta & \text{in } \Omega_1 \setminus \Omega_{12} \\
q_2^\Delta & \text{in } \Omega_2 \setminus \Omega_{12} \end{cases}
\]
which satisfy the Stokes problem
\[
-\text{div} \, T(w, q) = 0 \quad \text{in } \Omega
\]
\[
d\text{div} \, w = 0 \quad \text{in } \Omega
\]
\[
w = 0 \quad \text{on } \Gamma_D
\]
\[
T(w, q) \cdot n = 0 \quad \text{on } \Gamma_N
\]
whose unique solution is \( w = 0 \) and \( q = 0 \). Thus, we can conclude that \( w_1^\Delta = 0 \) in \( \Omega_1 \) \( (i = 1, 2) \) and \( u_1^{\lambda_1} = u_2^{\lambda_2} \) on \( \Gamma_1 \cup \Gamma_2 \).

Applying a similar argument to the state equations (58) with \( f = 0 \) and defining \( \tilde{w} = w_1^\Delta|_{\Omega_1} - w_2^\Delta|_{\Omega_2} \) and \( \tilde{q} = q_1|_{\Omega_1} - q_2|_{\Omega_2} \) in \( \Omega_{12} \), we can prove that both these functions are null and we can conclude that \( \lambda_i = 0 \), \( i = 1, 2 \).

If \( f \neq 0 \), for \( i = 1, 2 \) and \( j = 3 - i \), let \( w_i^f, q_i^f \) be the solution of the problem
\[
-\text{div} \, T(w_i^f, q_i^f) = 0 \quad \text{in } \Omega_i
\]
\[
d\text{div} \, w_i^f = 0 \quad \text{in } \Omega_i
\]
\[
w_i^f = u_i^0 - f_j \quad \text{on } \Gamma_i
\]
\[
w_i^f = 0 \quad \text{on } \Gamma_D
\]
\[
T(w_i^f, q_i^f) \cdot n = 0 \quad \text{on } \Gamma_N^i
\]
\[
\text{where, } u_i^0 \text{ being the solutions of (58) with } \lambda_i = 0. \text{ Then, we can write (59) as}
\[
(\Lambda^D)'(\chi(\Lambda), \mu')_{\Lambda^D} = - (\Lambda^D)'(A_f, \mu)_{\Lambda^D} \quad \forall \mu \in \Lambda^D,
\]
where
\[
A_f : \Lambda^D \rightarrow (\Lambda^D)'
\]
\[
(\Lambda^D)'(A_f, \mu)_{\Lambda^D} = \int_{\Gamma_1} ((u_1^{0,f} - u_2^{0,f}) + w_2^f) \mu_1 \, d\Gamma + \int_{\Gamma_2} (u_1^{0,f} - u_2^{0,f}) + w_2^f \mu_2 \, d\Gamma.
\]

The thesis follows from the same arguments used before. \[\square\]

Since the space \( \Lambda^D_N \) of discrete Dirichlet controls is a subset of \( \Lambda^D \), Lemma 5.1 Theorem 5.1 and Proposition 5.1 hold in the discrete case too and we can conclude that the minimization problem (16)–(17), or, equivalently, the optimality system (18)–(29), has a unique solution.

The minimum of the cost functional \( J_l \) is zero thanks to Proposition 2.1.

The real value \( \inf_{\lambda \in A^D_N} J_l(\lambda) \) attained at convergence, and reported in the second column of the Tables 1 and 11 is about \( \epsilon^2, \epsilon = 10^{-9} \) being the tolerance in the stopping criterion of Bi-CGSTab iterations. We notice that reducing the tolerance \( \epsilon \), \( \inf_{\lambda \in A^D_N} J_l(\lambda) \) reduces too. The errors between the discrete states \( (u_i, h, p_i, h) \) and the exact ones \( (u_i, p_i) \) vanish for \( h \to 0 \) and increasing \( p \) according to the theoretical convergence rate of \( h^p \)–finite element approximation.

5.2. Analysis of the optimal control problem with Neumann controls

For \( i = 1, 2 \), let
\[
\Lambda_i^N = [H^{-1/2}(\Gamma_i)]^d
\]

denote the spaces of admissible Neumann controls and we set
\[
\Lambda^N = \Lambda_1^N \times \Lambda_2^N.
\]
Let us denote Lemma 5.2 and the related norm can be defined through a Neumann to Dirichlet map defined any \( \phi \in H^{1/2}(\Gamma_i) \), \( \psi \in H^{1/2}(\Gamma_i) \), and the following we will still denote the completion of \( \Lambda_N^i \) with suitable homogeneous boundary conditions on \( \partial \Omega_i \setminus \Gamma_i \). The unknown controls on the interface are obtained by solving the minimization problem

\[
\inf_{\Delta \in \Lambda^N} \left[ \frac{1}{2} \left\| T(u_1^{\lambda_1}, p_1^{\lambda_1}) - n \right\|_{H^{1/2}(\Omega_1)}^2 + \frac{1}{2} \left\| T(u_2^{\lambda_2}, p_2^{\lambda_2}) - n \right\|_{H^{1/2}(\Omega_2)}^2 \right]
\]

Denoting by \( -\Delta_i \), the Laplace-Beltrami operator on \( \Gamma_i \), for any \( \psi, \phi \in H^{-1/2}(\Gamma_i) \) we define the following inner product (see, e.g., [12]):

\[
(\psi, \phi)_{H^{-1/2}(\Gamma_i)} = \int_{\Gamma_i} (-\Delta_i)^{-1/2} \psi \phi d\Gamma
\]

and the related norm \( \| \psi \|_{H^{-1/2}(\Gamma_i)} = (\psi, \psi)_{H^{-1/2}(\Gamma_i)}^{1/2} \).

The fractional Laplace-Beltrami operator \( (-\Delta_i)^{-1/2} \) can be defined through a Neumann to Dirichlet map defined from \( H^{-1/2}(\Gamma_i) \) to \( H^{1/2}(\Gamma_i) \) (see, e.g., [13]). Precisely, for any \( \phi \in H^{1/2}(\Gamma_i) \) we solve the problem

\[
\begin{cases}
-\Delta u + u = 0 & \text{in } \Omega_i \\
\partial u \bigg|_{\partial \Omega_i \setminus \Gamma_i} = 0 & \text{on } \partial \Omega_i \setminus \Gamma_i \\
\partial u \bigg|_{\Gamma_i} = \phi & \text{on } \Gamma_i
\end{cases}
\]

and we set \( (-\Delta_i)^{-1/2} \phi = u \bigg|_{\Gamma_i} \).

From now on, let \( \cdot, \cdot \bigg|_{H^{-1/2}(\Gamma_i, L^2)} \) and \( \| \cdot \|_{H^{-1/2}(\Gamma_i, L^2)} \) (see, e.g., [13]). Equations (65), (66) define an optimal control problem where both the control functions and the observations are of boundary (interface) type.

As for Dirichlet case, thanks to the linearity of the problem, we can equivalently express the cost functional as

\[
J_f(\Delta) = \frac{1}{2} \left\| T(u_1^{\lambda_1}, p_1^{\lambda_1}) - n \right\|_{H^{1/2}(\Omega_1)}^2 + \frac{1}{2} \left\| T(u_2^{\lambda_2}, p_2^{\lambda_2}) - n \right\|_{H^{1/2}(\Omega_2)}^2 + \frac{1}{2} \left( T(u_1^{0,f}, p_1^{0,f}) - n \right) \cdot \left( T(u_2^{0,f}, p_2^{0,f}) - n \right)_{-1/2}.
\]

Let us denote

\[
\|\Delta\|_N = \left\| T(u_1^{\lambda_1}, p_1^{\lambda_1}) - n \right\|_{H^{1/2}(\Omega_1)} + \left\| T(u_2^{\lambda_2}, p_2^{\lambda_2}) - n \right\|_{H^{1/2}(\Omega_2)}.
\]

\[\text{Lemma 5.2} \iff \partial \Omega_i \cap \Gamma_D \neq \emptyset, \text{ then } \|\Delta\|_N \text{ defines a norm on the space } \Lambda_N^i.\]

\[\text{Proof.} \quad \text{We proceed as done for Dirichlet controls: } \|\Delta\|_N \text{ is always a semi-norm on } \Lambda_N^i, \text{ we only have to prove that, if } \|\Delta\|_N = 0, \text{ then } \Delta = 0. \text{ Obviously, } \|\Delta\|_N = 0 \implies T(u_1^{\lambda_1}, p_1^{\lambda_1}) \cdot n = T(u_2^{\lambda_2}, p_2^{\lambda_2}) \cdot n \text{ a.e. on } \Omega_1 \cup \Gamma_2. \text{ In view of Proposition 3.2 starting from } (u_1^{\lambda_1}, p_1^{\lambda_1}) \text{ we define the pair } (\nabla, \Psi) \text{ as in (63), (64), that satisfies a Stokes problem in } \Omega \text{ with null force and homogeneous boundary conditions. This problem is well-posed and, in particular, } \nabla = 0 \text{ and } \Psi = 0 \text{ a.e. in } \Omega. \text{ This implies that } T(\nabla, \Psi) \cdot n = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \text{ and, for } i = 1, 2, \lambda_i = 0 \text{ in } \Lambda_N^i.\]

We cannot guarantee that \( \Lambda_N^i \) is complete with respect to the norm \( \|\Delta\|_N \), but we can construct its completion, say \( \tilde{\Lambda}^i \), with respect to such norm. For the sake of notation, in the following we will still denote the completion of \( \Lambda_N^i \) by the same symbol.

\[\text{Theorem 5.2 Consider the minimization problem}
\]

\[
\inf_{\Delta \in \tilde{\Lambda}^N} J_f(\Delta).
\]

If \( \partial \Omega_{i2} \cap \Gamma_D \neq \emptyset, \text{ problem } (69) \text{ has a unique solution satisfying}
\]

\[
(\Lambda^N, (\tilde{J}_f(\Delta), \mu^N) =

(\lambda_1, \lambda_2) =

\begin{cases} 
T(u_1^{\lambda_1}, p_1^{\lambda_1}) \cdot n & T(u_2^{\lambda_2}, p_2^{\lambda_2}) \cdot n, \\
\left( T(u_1^{0,f}, p_1^{0,f}) \cdot n - T(u_2^{0,f}, p_2^{0,f}) \cdot n \right)_{-1/2} = 0
\end{cases}
\]

for all \( \mu \in \tilde{\Lambda}^N. \)

\[\text{Proof.} \quad \text{The proof follows the same guidelines of the proof of Theorem 5.1.}\]

In view of (65), the Euler-Lagrange equation (70) becomes:

\[
\sum_{i=1}^{2} \int_{\Gamma_i} (-\Delta_i)^{-1/2} \left( T(u_1^{\lambda_1}, p_1^{\lambda_1}) \cdot n - T(u_2^{\lambda_2}, p_2^{\lambda_2}) \cdot n \right) \left( T(u_1^{0,f}, p_1^{0,f}) \cdot n - T(u_2^{0,f}, p_2^{0,f}) \cdot n \right) d\Gamma = 0
\]

for all \( \mu \in \tilde{\Lambda}^N \) and \( j = 3 - i. \)

Solving equation (71) is equivalent to solving the following optimality system: find \( \tilde{\Delta} = (\lambda_1, \lambda_2) \in \tilde{\Lambda}^N \) and, for \( i = 1, 2, (u_i, p_i) \in V_{i,0} \times Q_i, (w_i, q_i) \in V_{i,0} \times Q_i \) such that

\[
\begin{align*}
-\text{div } T(u_i, p_i) &= f \text{ in } \Omega_i \\
\text{div } u_i &= 0 \text{ in } \Omega_i \\
T(u_i, p_i) \cdot n &= \lambda_i \text{ on } \Gamma_i \\
T(u_i, p_i) \cdot n &= 0 \text{ on } \Gamma_N
\end{align*}
\]

(72)
The optimality system (72)-(74) in C. Canuto, P. Gervasio, and A. Quarteroni. Finite-element methods... obviously more attractive than (75). Indeed, the solution satisfies (74) which implies that \( T(u_1^{\lambda_1}, f \mid p_1^{\lambda_1}, f) \cdot n = T(u_2^{\lambda_2}, f \mid p_2^{\lambda_2}, f) \cdot n \) on \( \Gamma_1 \cup \Gamma_2 \). As a consequence the solutions \( (\bar{w}_i, q_i) \) of (73) are identically null and (74) is satisfied.

To prove that this solution is unique, we proceed as in the proof of Proposition 5.1 by exploiting linearity, continuity and coercivity of the Laplace-Beltrami operator (see [67]).

Proposition 5.2 The optimality system (72)-(74) has a unique solution whose control component \( \lambda \in \Lambda_N \) is the solution of the Euler-Lagrange equation (71).

Proof. Let \( \lambda \) be the solution of (75). Theorem 5.2 guarantees that such solution exists and is unique. Then, it is also a solution of (72)-(74). Indeed, the solution satisfies (71) which implies that \( T(u_1^{\lambda_1}, f \mid p_1^{\lambda_1}, f) \cdot n = T(u_2^{\lambda_2}, f \mid p_2^{\lambda_2}, f) \cdot n \) on \( \Gamma_1 \cup \Gamma_2 \). As a consequence the solutions \( (\bar{w}_i, q_i) \) of (73) are identically null and (74) is satisfied.

To prove that this solution is unique, we proceed as in the proof of Proposition 5.1 by exploiting linearity, continuity and coercivity of the Laplace-Beltrami operator (see [67]).

6. CONCLUSIONS

We have studied the ICDD method for the mathematical formulation and the numerical solution of the Stokes problem. This method rests on the reformulation of the original boundary value problem as an optimal control problem involving control variables that represent the trace of the velocity or the normal stress across the subdomain interfaces. We have shown that choosing control variables of mixed type allows to set up a robust numerical method with convergence rate independent of the discretization parameters as well as of the size of the overlapping region. Possible extensions of this work could consider the case of decomposition with more than two subdomains and heterogeneous couplings like, e.g., the Stokes/Darcy problem (see [6]).

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