Anisotropic finite elements for the transient transport equation

Yves Bourgault and Marco Picasso
ANISOTROPIC FINITE ELEMENTS FOR THE TRANSIENT TRANSPORT EQUATION

YVES BOURGAULT∗ AND MARCO PICASSO†

Abstract. A stabilized finite element discretization of the transient transport equation is studied in the framework of anisotropic meshes. A priori and a posteriori error estimates are derived, the involved constants being independent of the mesh aspect ratio. Numerical results on non adapted, anisotropic meshes confirm the sharpness of the theoretical predictions. An anisotropic, adaptive finite element algorithm is then proposed with goal to control the $L^2$ error in space at final time. Numerical results are then presented on anisotropic, adapted meshes.

1. Introduction. A posteriori error estimates and adaptive finite element algorithms are now widely used for elliptic and parabolic problems. However, fewer contributions are available for hyperbolic problems.

A posteriori error estimates for hyperbolic problems of second order have been considered for instance in [6, 5, 20]. In [14, 8], a posteriori error estimates for the stationary transport-reaction equation are derived. In [17], a review of a posteriori error estimates and adaptive finite elements is proposed for nonlinear conservation laws, using finite volumes or discontinuous Galerkin finite elements.

In this paper, a priori and a posteriori error estimates are proved for a semi-discrete finite elements discretization of the transient transport equation. The analysis remains valid with anisotropic meshes - that is to say meshes with large aspect ratio, up to 500 in practice - in the sense that the constants involved in the error bounds do not depend on the mesh aspect ratio.

Numerical experiments are performed using the Crank-Nicolson time discretization with small time steps, so that the error due to time discretization remains negligible. The effectivity index remain between 1 and 5, even with meshes having aspect ratio 500! It should be noted that the full error analysis including time and space discretization is not included in this paper. The second author has performed such an analysis for the heat equation [15], but this is beyond the scope of the present paper.

An adaptive finite element algorithm is then proposed with goal to control the $L^2$ error at final time. Accurate results are obtained, nevertheless, the method used to interpolate the computed solution between two successive adapted meshes is shown to be critical.

2. Problem setting. Let $T > 0$, let $\Omega$ be a domain of $\mathbb{R}^2$ with polygonal boundary $\partial \Omega$ and unit outer normal $n$. Let $\beta \neq 0$, $\beta \in C^1(\bar{\Omega})$ with $\text{div} \beta = 0$ in $\Omega$ and $\beta \cdot n = 0$ on $\partial \Omega$. Given $f \in L^2(0,T; L^2(\Omega))$ and $u_0 \in L^2(\Omega)$, we are searching for $u$ satisfying $u(0) = u_0$ and

$$\frac{\partial u}{\partial t} + \beta \cdot \nabla u = f \quad \text{in } \Omega \times (0,T),$$

(2.1)

Throughout this paper, when necessary, it will be assumed that the data $\Omega$, $\beta$, $f$ and $u_0$ are such that the solution $u$ to the above problem exists and is sufficiently smooth.

The following finite element scheme is considered. For any $h > 0$, let $\mathcal{T}_h$ be a conforming mesh of $\Omega$ into triangles $K$ with diameter $h_K$ less than $h$. Let $V_h$ be the
usual finite element space of continuous functions that are linear on the triangles of $\mathcal{T}_h$. Let $r_h : C(\Omega) \to V_h$ be the usual Lagrange interpolant, assuming $u_0 \in C(\Omega)$, we are looking for $u_h : (0,T) \to V_h$ such that $u_h(0) = r_h u_0$ and for all $0 \leq t \leq T$:

$$\int_{\Omega} \left( \frac{\partial u_h}{\partial t} + \beta \cdot \nabla u_h - f \right) \left( v_h + \delta_h \beta \cdot \nabla v_h \right) \, dx = 0 \quad \forall v_h \in V_h. \quad (2.2)$$

It has been proved in [9] that the backward Euler scheme corresponding to the above finite element scheme is convergent with a constant stabilization parameter

$$\delta_h = \frac{h}{2|\beta'|},$$

where $|\beta'|$ is the average velocity. In this paper, anisotropic meshes are considered, that is to say meshes with large aspect ratio, up to 500 in practice, and the following choice is advocated

$$\delta_h|_K = \frac{\lambda_{2,K}}{2|\beta'| L^\infty(K)^2} \quad \forall K \in \mathcal{T}_h. \quad (2.3)$$

Here $\lambda_{2,K}$ denotes the mesh size in the direction of minimum stretching. Note that this choice is compatible with that of [16] in the framework of convection dominated, stationary convection-diffusion problems. Also note that $\lambda_{2,K}$ can be set to $h_K$ on isotropic meshes.

We now clarify the definition of $\lambda_{2,K}$ in (2.3) and follow the framework of [12, 13]. For any triangle $K$, let $T_K : \hat{K} \to K$ be the affine transformation which maps the reference triangle $\hat{K}$ into $K$. Let $M_K$ be the Jacobian of $T_K$ that is

$$x = T_K(\hat{x}) = M_K \hat{x} + t_K.$$

Since $M_K$ is invertible, it admits a singular value decomposition $M_K = R_K^T \Lambda_K P_K$, where $R_K$ and $P_K$ are orthogonal and where $\Lambda_K$ is diagonal with positive entries. In the following we set

$$\Lambda_K = \begin{pmatrix} \lambda_{1,K} & 0 \\ 0 & \lambda_{2,K} \end{pmatrix} \quad \text{and} \quad R_K = \begin{pmatrix} r_{1,K} \\ r_{2,K} \end{pmatrix}, \quad (2.4)$$

with the choice $\lambda_{1,K} \geq \lambda_{2,K}$. The unit vectors $r_{1,K}, r_{2,K}$ correspond to the directions of maximum and minimum stretching and $\lambda_{1,K}, \lambda_{2,K}$ are the values of maximum and minimum stretching.

In the frame of anisotropic meshes, the classical minimum angle condition is not required. However, since Clément’s interpolant will be used [11], the following, technical restrictions, hold. For each vertex, the number of neighbouring vertices should be bounded from above, uniformly with respect to the mesh size $h$. Also, for each triangle $K$ of the mesh, there is a restriction related to the patch $\Delta_K$, the set of triangles having a vertex common with $K$. More precisely, the diameter of the reference patch $\Delta_{\hat{K}}$, that is $\Delta_{\hat{K}} = T_K^{-1}(\Delta_K)$, must be uniformly bounded independently of the mesh geometry. We refer to [16] for details.

3. A priori error estimates. We now prove that the solution of (2.2) converges to that of (2.1). The main idea is taken from [9] and consists in choosing $u_h + \delta_h \frac{\partial u_h}{\partial t}$ in (2.2) as test function. This is possible provided the stabilization parameter $\delta_h$ is constant in space and time.
Theorem 3.1. Let \( u \) be the solution of (2.1), let \( u_h \) be the solution of (2.2) with \( \delta_h \) being defined by

\[
\delta_h = \max_{K \in T_h} \frac{\lambda_{2,K}}{2\|\beta\|_{L^\infty(\Omega)^2}},
\]

instead of (2.3). Assume that \( \Omega, \beta, f \) and \( u_0 \) are such that \( u \in H^1(0,T;H^2(\Omega)) \) and let \( e = u - u_h \). Then, there exists \( C \) independent of \( \beta, u \), the mesh size and aspect ratio such that

\[
\frac{d}{dt} \int_\Omega e^2 dx + \int_\Omega \delta_h \left( \frac{\partial e}{\partial t} + \beta \cdot \nabla e \right)^2 dx + \frac{d}{dt} \int_\Omega \delta_h^2 (\beta \cdot \nabla e)^2 dx \\
\leq C \sum_{K \in T_h} \left( \frac{1}{\delta_h} + \frac{\delta_h \|\beta\|^2_{L^\infty(K)^2}}{\lambda_{2,K}} \right) L^2_K(u) \\
+ \left( \delta_h + \frac{\delta_h \|\beta\|^2_{L^\infty(K)^2}}{\lambda_{2,K}^2} \right) L^2_K \left( \frac{\partial u}{\partial t} \right).
\]  

(3.1)

Here we have denoted, for any \( v \in H^2(\Omega) \)

\[
L^2_K(v) = \lambda_{1,K}^4 \int_K (r_{1,K}^T H(v) r_{1,K})^2 \\
+ \lambda_{1,K}^2 \lambda_{2,K}^2 \int_K (r_{2,K}^T H(v) r_{2,K})^2 \\
+ \lambda_{2,K}^4 \int_K (r_{2,K}^T H(v) r_{2,K})^2,
\]

where \( H(v) \) is the Hessian matrix defined by

\[
H(v) = \begin{pmatrix}
\frac{\partial^2 v}{\partial x_1^2} & \frac{\partial^2 v}{\partial x_1 \partial x_2} \\
\frac{\partial^2 v}{\partial x_2 \partial x_1} & \frac{\partial^2 v}{\partial x_2^2}
\end{pmatrix}.
\]

Remark 3.2. In the case of isotropic meshes, \( \lambda_{1,K} \simeq \lambda_{2,K} \simeq h_K \), \( L^2_K(u) \leq Ch^4_{H^2(K)} \|u\|_{H^2(K)}^2 \), where \( C \) is independent of the mesh size but depends on the aspect ratio. Thus estimate (3.1) yields, after time integration from 0 to \( T \), to the classical \( O(h^{3/2}) \) estimate:

\[
\int_\Omega e^2(T) dx \leq Ch^3 + h.o.t.,
\]

where \( C \) is independent of the mesh size but depends on the aspect ratio and where h.o.t. denotes higher order terms.

Remark 3.3. Estimate (3.1) is optimal for anisotropic meshes in the following sense. Assume for instance that \( u \) depends only on \( x_2 \) and that the mesh is aligned with the solution, that is \( r_{1,K} = (1,0)^T, r_{2,K} = (0,1)^T \), then (3.1) becomes, since
\[
\begin{aligned}
r_{T,K}^H(u) r_{1,K} = r_{T,K}^H(u) r_{2,K} = 0 : \\
\frac{d}{dt} \int_\Omega e^2 dx + \int_\Omega \delta_h \left( \frac{\partial e}{\partial t} + \beta \cdot \nabla e \right)^2 dx + \frac{d}{dt} \int_\Omega \delta_h^2 \beta \cdot \nabla e^2 dx \\
\leq C \sum_{K \in T_h} \left( \max_{K \in T_h} \lambda_{2,K} \right)^3 \|\beta\|_{L^\infty(\Omega)}^3 \int_\Omega \left( \frac{\partial^2 u}{\partial x_2^2} \right)^2 dx \\
+ \left( \max_{K \in T_h} \lambda_{2,K} \right)^5 \|\beta\|_{L^\infty(\Omega)} \int_\Omega \left( \frac{\partial^2 \partial e}{\partial x_2^2 \partial t} \right)^2 dx.
\end{aligned}
\]

Therefore, convergence is achieved when \( \max_{K \in T_h} \lambda_{2,K} \to 0 \), no matter what \( \lambda_{1,K} \) (and thus the aspect ratio) is.

**Proof.** We formally write (2.2) as

\[
B(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h,
\]

so that the error \( e = u - u_h \) satisfies

\[
B(e, v_h) = 0 \quad \forall v_h \in V_h. \tag{3.2}
\]

We then remark that, for all \( v \in H^1(0, T; H^1(\Omega)) \), we have

\[
B \left( v + \delta_h \frac{\partial v}{\partial t}, v \right) = \int_\Omega \left( \frac{\partial v}{\partial t} + \beta \cdot \nabla v \right) \left( v + \delta_h \left( \frac{\partial v}{\partial t} + \beta \cdot \nabla v \right) + \delta_h^2 \beta \cdot \nabla \frac{\partial v}{\partial t} \right) dx \\
= \frac{1}{2} \frac{d}{dt} \int_\Omega v^2 dx + \int_\Omega \delta_h \left( \frac{\partial v}{\partial t} + \beta \cdot \nabla v \right)^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \delta_h^2 \beta \cdot \nabla v^2 dx,
\]

where we have used the fact that

\[
\int_\Omega (\beta \cdot \nabla v) v dx = 0 \quad \forall v \in H^1(\Omega). \tag{3.3}
\]

Choosing \( v = e \) and using (3.2), we therefore have, for all \( v_h \in V_h \)

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega e^2 dx + \int_\Omega \delta_h \left( \frac{\partial e}{\partial t} + \beta \cdot \nabla e \right)^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \delta_h^2 \beta \cdot \nabla e^2 dx \\
= B \left( e, e + \delta_h \frac{\partial e}{\partial t} \right) \\
= B \left( e, u - v_h + \delta_h \frac{\partial}{\partial t} (u - v_h) \right) \\
= \int_\Omega \left( \frac{\partial e}{\partial t} + \beta \cdot \nabla e \right) \left( u - v_h + \delta_h \left( \frac{\partial}{\partial t} (u - v_h) + \beta \cdot \nabla (u - v_h) \right) \\
+ \delta_h^2 \beta \cdot \nabla \frac{\partial}{\partial t} (u - v_h) \right) dx.
\]
Young’s inequality yields
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} e^2 dx + \int_{\Omega} \delta_h \left( \frac{\partial e}{\partial t} + \beta \cdot \nabla e \right)^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \delta_h (\beta \cdot \nabla e)^2 dx \\
\leq \frac{1}{2} \int_{\Omega} \delta_h \left( \frac{\partial e}{\partial t} + \beta \cdot \nabla e \right)^2 dx \\
+ \frac{4}{2\delta_h} \left( \|u - v_h\|_{L^2(\Omega)}^2 + \delta_h^2 \left\| \frac{\partial}{\partial t} (u - v_h) \right\|_{L^2(\Omega)}^2 \\
+ \delta_h^2 \|\beta \cdot \nabla (u - v_h)\|_{L^2(\Omega)}^2 + \delta_h^4 \left\| \beta \cdot \nabla \frac{\partial}{\partial t} (u - v_h) \right\|_{L^2(\Omega)}^2 \right).
\]

We now choose \(v_h = r_h u\), the Lagrange interpolant of \(u\), the interpolation results of Lemma 2 in [13]: there exists \(C\) independent of the mesh size and aspect ratio such that, for all \(v \in H^2(\Omega)\)
\[
\|v - r_h v\|_{L^2(K)}^2 + \lambda_{2,K}^2 \|\nabla (v - r_h v)\|_{L^2(K)}^2 \leq C L_K^2(v),
\]
to obtain (3.1).

4. A posteriori error estimate. **Theorem 4.1.** Let \(u\) be the solution of (2.1), let \(u_h\) be the solution of (2.2) with \(\delta_h\) being defined by (2.3). Assume that \(\Omega, \beta, f\) and \(u_0\) are such that \(u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))\), let \(e = u - u_h\). Then, there exists \(C\) independent of \(\beta, u\), the mesh size and aspect ratio such that
\[
\int_{\Omega} e(T)^2 dx \leq \int_{\Omega} e(0)^2 dx + C \sum_{K \in T_h} \int_0^T \left\| f - \frac{\partial u_h}{\partial t} - \beta \cdot \nabla u_h \right\|_{L^2(K)}^2 \omega_K(e). \tag{4.1}
\]
Here \(\omega_K(e)\) is defined for all \(v \in H^1(\Omega)\) by
\[
\omega_K^2(v) = \lambda_{1,K}^2 (r_{1,K}^T G_K(v) r_{1,K}) + \lambda_{2,K}^2 (r_{2,K}^T G_K(v) r_{2,K}),
\]
where \(G_K(v)\) denotes the \(2 \times 2\) matrix defined by
\[
G_K(v) = \left( \begin{array}{cc} \int_{\Delta_K} \left( \frac{\partial v}{\partial x_1} \right)^2 dx & \int_{\Delta_K} \frac{\partial v}{\partial x_1} \frac{\partial v}{\partial x_2} dx \\ \int_{\Delta_K} \frac{\partial v}{\partial x_1} \frac{\partial v}{\partial x_2} dx & \int_{\Delta_K} \left( \frac{\partial v}{\partial x_2} \right)^2 dx \end{array} \right). \tag{4.2}
\]

**Remark 4.2.** Estimate (4.1) is not standard in the a posteriori sense since \(e\) - and thus \(u\) - is involved in the right hand side. However, \(\omega_K(e)\) can be estimated using post-processing techniques as in [18, 19] which leads to sharp error indicators, at least for elliptic and parabolic problems. These post-processing techniques can be justified whenever superconvergence occurs, see for instance [1, 21] for elliptic problems and [22] for the stationary transport equation. The numerical results presented at the end of this paper show that such techniques can also be applied to the time-dependent transport equation with stabilized finite elements and anisotropic meshes.
Proof. Using (2.1), (2.2) and Cauchy-Schwarz inequality, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} e^2 dx = \int_{\Omega} \left( \frac{\partial e}{\partial t} e + (\beta \cdot \nabla e) \right) dx
\]

\[
= \int_{\Omega} \left( f - \frac{\partial u_h}{\partial t} - \nabla u_h \right) e dx
\]

\[
= \int_{\Omega} \left( f - \frac{\partial u_h}{\partial t} - \nabla u_h \right) \left( e - v_h - \delta_h \beta \cdot \nabla v_h \right) dx
\]

\[
\leq \sum_{K \in T_h} \left\| f - \frac{\partial u_h}{\partial t} - \nabla u_h \right\|_{L^2(K)} \left( \|e - R_h e\|_{L^2(K)} + \|\delta_h \beta \cdot \nabla R_h e\|_{L^2(K)} \right).
\]

(4.3)

for all \( v_h \in V_h \). We then choose \( v_h = R_h e \), where \( R_h \) is Clément’s interpolant [11]. From Proposition 3.2 in [12], there exists a constant \( C \) depending only on the reference element \( K \) such that, for all \( v \in H^1(\Omega) \), for all \( K \in T_h \)

\[
\|v - R_h v\|_{L^2(K)}^2 + \lambda_{2,K}^2 \|\nabla (v - R_h v)\|_{L^2(K)}^2 \leq C \omega_K^2(v).
\]

(4.4)

Using the triangle inequality, (2.3) and (4.4) we obtain

\[
\|e - R_h e\|_{L^2(K)} + \|\delta_h \beta \cdot \nabla R_h e\|_{L^2(K)} \leq C \left( \omega_K(e) + \lambda_{2,K} \|\nabla e\|_{L^2(\Delta_K)} \right).
\]

Since \( \nabla e = (\nabla e \cdot r_{1,K}) r_{1,K} + (\nabla e \cdot r_{2,K}) r_{2,K} \) and since

\[
\|\nabla e \cdot r_{i,K}\|_{L^2(\Delta_K)}^2 = r_{i,K}^T G_K(e) r_{i,K} \quad i = 1, 2,
\]

then

\[
\|e - R_h e\|_{L^2(K)} + \|\delta_h \beta \cdot \nabla R_h e\|_{L^2(K)} \leq C \omega_K(e).
\]

The above estimate in (4.3) yields, after time integration, to (4.1). \( \square \)

5. Time discretization. For any positive integer \( N \), let \( \tau = T/N \) be the time step, \( t^n = n \tau \), \( n = 0, 1, 2, \ldots, N - 1 \). The Crank-Nicolson scheme is considered to discretize (2.2). More precisely, starting from \( u_h^0 = r_h u_0 \), for each \( n = 0, 1, 2, \ldots, N - 1 \), given \( u_h^n \in V_h \), then we are looking for \( u_h^{n+1} \in V_h \) such that

\[
\int_{\Omega} \left( \frac{u_h^{n+1} - u_h^n}{\tau} + \frac{1}{2} \beta \cdot \nabla (u_h^n + u_h^{n+1}) - \frac{1}{2} \left( f(t^n) + f(t^{n+1}) \right) \left( v_h + \delta_h \beta \cdot \nabla v_h \right) dx = 0 \quad \forall v_h \in V_h,
\]

(5.1)

where we have assumed that \( f \in \mathcal{C}^0([0,T];L^2(\Omega)) \). We then define \( u_{h,T} \) as

\[
u_{h,T}(x,t) = \frac{t - t^n}{\tau} u_h^{n+1} + \frac{t^{n+1} - t}{\tau} u_h^n \quad \forall(x,t) \in \bar{\Omega} \times (t^n, t^{n+1})
\]

and use

\[
\sum_{K \in T_h} \int_0^T \eta_K^2
\]

as error estimator for

\[
\int_{\Omega} (u - u_{h,T})^2(T)
\]

6
where $\eta^2_K$ is defined by

$$
\eta^2_K = \| f - \frac{\partial u_{h\tau}}{\partial t} - \beta \cdot \nabla u_{h\tau} \|_{L^2(K)} \omega_K(u - u_{h\tau}),
$$

and where $\omega_K(u - u_{h\tau})$ is approached using ZZ post-processing as in [18, 19]. More precisely, we replace the derivatives

$$
\frac{\partial (u - u_h)}{\partial x_i} \text{ in (4.2) by } \Pi_h \frac{\partial u_h}{\partial x_i} - \frac{\partial u_h}{\partial x_i}, \ i = 1, 2,
$$

where $\Pi_h$ is the approximate $L^2(\Omega)$ projection onto $V_h$ defined for each vertex $P$ of $T_h$ by

$$
\begin{bmatrix}
\Pi_h \left( \frac{\partial u_h}{\partial x_1} \right)(P) \\
\Pi_h \left( \frac{\partial u_h}{\partial x_2} \right)(P)
\end{bmatrix} = \frac{1}{\sum_{K \in T_h} |K|} \sum_{K \in T_h} |K| \begin{bmatrix}
\frac{\partial u_h}{\partial x_1} \\
\frac{\partial u_h}{\partial x_2}
\end{bmatrix}.
$$

Since the Crank-Nicolson scheme is order two in time, it is expected that

$$
\| (u - u_{h\tau})(T) \|_{L^2(\Omega)} = O(h^{3/2} + \tau^2).
$$

Thus, when $\tau = O(h^2)$, the error due to time discretization is asymptotically negligible so that $u - u_{h\tau}$ converges to the error due to space discretization only, namely $u - u_h$.

6. **Numerical experiments with non adapted meshes.** We consider the case when $\Omega = (0, 1)^2$, $T = 0.5$, $\beta = (1, 0)^T$, $f = 0$, the initial condition is a smoothed step function

$$
u_0(x_1, x_2) = \tanh \left( -C((x_1 - 0.25)^2 - 0.01) \right),
$$

thus $u(x_1, x_2, t) = u_0(x_1 - t, x_2)$. The solution is smooth, with value close to $\pm 1$, except in an internal layer of small width, driven by the parameter $C$. $C = 60$ corresponds to a rather wide layer width while $C = 240$ corresponds to a small one. Zero Dirichlet boundary conditions apply along the left side, whereas no boundary conditions apply on the other sides. Note that this test case does not fit the assumption $\beta \cdot n = 0$ on the boundary therefore (3.3) does not hold but, since

$$
\int_{\Omega} (\beta \cdot \nabla v)v dx = \frac{1}{2} \int_{\partial \Omega} (\beta \cdot n)v^2 ds \geq 0,
$$

for all $v \in H^1(\Omega)$, $v = 0$ along the left side, then the error estimates are still valid.

The numerical solution computed on an anisotropic mesh with typical meshsize $h_1 = 0.01$, $h_2 = 0.5$ (aspect ratio 50) and time step $\tau = 0.002$ is reported in Fig. 6.1.
Fig. 6.1. Solution at initial and final time when \( h_1 = 0.01, h_2 = 0.5 \) (aspect ratio 50), \( \tau = 0.002 \). Top row \( C = 60 \), bottom row \( C = 240 \).

Numerical experiments have been performed with anisotropic, non adapted meshes with aspect ratio varying from 50 to 500, generated with the BL2D software [7]. The quality of our error estimator is investigated, the notations used being reported in Table 6.1. Numerical results are reported in Table 6.2 and 6.3. The time step is taken to be \( \tau = O(h^2) \) so that the error due to time discretization is asymptotically negligible. There is no clear convergence rate, the numerical results show that the error in the \( L^2(0, T; H^1(\Omega)) \) norm is \( \simeq O(h^{1.8}) \) while the error in the \( L^2(\Omega) \) norm at final time is \( \simeq O(h^{1.8}) \). However, it should be noted that the post-processed ZZ gradient seems to be asymptotically exact and that the effectivity index corresponding to our error anisotropic error converges to a value close to 5.
\[ h_1 - h_2 : \text{requested meshsize in directions } x_1 - x_2 \]

\[
e_{L^2(H^1)} = \left( \int_0^T \int_\Omega |\nabla (u - u_{h\tau})|^2 \right)^{1/2}
\]

\[
e_{izz} = \left( \int_0^T \int_\Omega |\nabla u_{h\tau} - \Pi_h \nabla u_{h\tau}|^2 \right)^{1/2}
\]

\[
e_{(T)_{L^2}} = \left( \int_\Omega (u - u_{h\tau})^2(T) \right)^{1/2}
\]

\[
e_{i_{ANI}} = \left( \int_0^T \sum_{K \in T_h} \eta_K^2 \right)^{1/2}
\]

**Table 6.1**

Notations.

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<th>( \tau )</th>
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<th>( e_{izz} )</th>
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<th>( e_{i_{ANI}} )</th>
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</table>

**Table 6.2**

Convergence results when \( \tau = O(h^2) \) and \( C = 60 \) on anisotropic meshes with aspect ratio 50 (rows 1-3) and 500 (rows 4-6).

<table>
<thead>
<tr>
<th>( h_1 - h_2 )</th>
<th>( \tau )</th>
<th>( e_{L^2(H^1)} )</th>
<th>( e_{izz} )</th>
<th>( e_{(T)_{L^2}} )</th>
<th>( e_{i_{ANI}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01-0.5</td>
<td>0.002</td>
<td>3.27</td>
<td>0.17</td>
<td>0.099</td>
<td>0.85</td>
</tr>
<tr>
<td>0.005-0.25</td>
<td>0.0005</td>
<td>1.86</td>
<td>0.21</td>
<td>0.046</td>
<td>0.89</td>
</tr>
<tr>
<td>0.0025-0.125</td>
<td>0.000125</td>
<td>0.75</td>
<td>0.30</td>
<td>0.016</td>
<td>1.08</td>
</tr>
<tr>
<td>0.001-0.5</td>
<td>0.002</td>
<td>0.45</td>
<td>0.18</td>
<td>0.0094</td>
<td>0.49</td>
</tr>
<tr>
<td>0.0005-0.25</td>
<td>0.0005</td>
<td>0.071</td>
<td>0.65</td>
<td>0.0011</td>
<td>1.79</td>
</tr>
<tr>
<td>0.00025-0.125</td>
<td>0.000125</td>
<td>0.028</td>
<td>0.92</td>
<td>0.00021</td>
<td>3.26</td>
</tr>
</tbody>
</table>

**Table 6.3**

Convergence results when \( \tau = O(h^2) \) and \( C = 240 \) on anisotropic meshes with aspect ratio 50 (rows 1-3) and 500 (rows 4-6).

7. **An adaptive algorithm.** We now present an adaptive algorithm based on our anisotropic error estimator. The goal is, given a time step \( \tau \), to build successive triangulations in order to control

\[ \|(u - u_{h\tau})(T)\|_{L^2(\Omega)}. \]
More precisely, given a preset tolerance $TOL$, we want to insure that

$$0.75 \; TOL \leq \left( \frac{\sum_{K \in T_h} \int_0^T \eta^2_K}{T} \right)^{1/2} \leq 1.25 \; TOL.$$ 

A sufficient condition is that, for each $n = 0, 1, 2, ..., N - 1$, we build a triangulation $T_h^n$ such that

$$0.75^2 \; TOL^2 \; \tau \leq \sum_{K \in T_h} \int_{t^n}^{t^{n+1}} \eta^2_K \leq 1.25^2 \; TOL^2 \; \tau.$$ 

In practice, a mesh satisfying the above inequalities is build by equidistributing $\eta_K$ in the directions $r_{1,K}$ and $r_{2,K}$ - the directions of maximum and minimum stretching and by aligning each triangle $K$ with the eigenvectors of the error gradient $G_K(e)$, the error gradient matrix post-processed using ZZ post-processing. All the meshes are generated using the BL2D software [7]. We refer to [18, 19] for details about the adaptive algorithm in the framework of elliptic and parabolic problems.

8. Numerical results with adapted meshes. The initial mesh is an isotropic mesh with typical meshsize $h = 0.1$. Unless specified, interpolation between two successive meshes is linear and provided by the BL2D software [7]. The mesh and numerical solution at initial and final time are reported in Fig. 8.1 and 8.2 when $C = 240$, $TOL = 0.01$, $\tau = 0.002$. Numerical results are reported when using several values of $TOL$ and $\tau = O(TOL^2)$ so that the error due to time discretization remains negligible. The additional notations used in this Section are reported in Table 8.1 and the results are shown in Table 8.2 when $C = 60$ and 8.3 when $C = 240$. Unlike what has been observed with non adapted meshes, the ZZ effectivity index does not converge to 1 when $TOL$ goes to zero, it remains below 0.75. As a consequence the effectivity index corresponding to our anisotropic error estimator remains below 2.3. Moreover, from the last two rows of Table 8.2, it is observed that the error increases from $TOL = 0.0025$ to $TOL = 0.00125$. We conjecture that this phenomenon is due to the error arising when interpolating the numerical solution between meshes, as already observed in [20] in the framework of the wave equation. Also note that this phenomenon was not observed for linear [18] and nonlinear parabolic problems [10]. In order to assess the importance of interpolation between meshes, numerical results are reported when using the conservative interpolation algorithm of [3], see Table 8.4 and 8.5. The numerical results are in favour of conservative interpolation, compare the last row of Table 8.2 and 8.4 ($C = 60$), or the last row of Table 8.3 and 8.5 ($C = 240$). Note that in [2], an alternative adaptive algorithm is proposed. The number of remeshings, say $M$, is prescribed by the user and the aim is to build an optimal adapted mesh for time intervals of length $T/M$. The interpolation error due to remeshing can then be controlled with the parameter $M$. 

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\[ N_v : \quad \text{number of vertices of the adapted mesh at final time.} \]
\[ ar : \quad \text{average aspect ratio, the aspect ratio on element } K \text{ being } \lambda_{1,K}/\lambda_{2,K}. \]
\[ N_{mesh} : \quad \text{number of adapted meshes when running the adaptive algorithm from initial to final time.} \]

**Table 8.1**

*Additional notations.*

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**Fig. 8.1.** Mesh and numerical solution at initial and final time when \( C = 240, \ TOL = 0.01, \tau = 0.002. \)

**Fig. 8.2.** Zoom at final time when \( C = 240, \ TOL = 0.01, \tau = 0.002. \)
The last test case is classical [4] in the framework of the Volume of Fluid method and corresponds to the stretching of a circle in a vortex flow. The initial condition corresponds to a smoothed step function

\[ u_0(x_1, x_2) = \exp \left( -C \left( \sqrt{(x_1 - 0.75)^2 + (x_2 - 0.15)^2} \right) \right), \]

with \( C = 60 \) or \( C = 240 \). The velocity field is given by

\[ \beta = \begin{pmatrix} -2 \sin(\Pi y) \cos(\Pi y) \sin^2(\Pi x) \\ 2 \sin(\Pi x) \cos(\Pi x) \sin^2(\Pi y) \end{pmatrix}, \quad 0 \leq t \leq 2, \]
it is reversed for $2 < t \leq 4$, so that the initial condition should be recovered at time $t = 4$, $u(x_1, x_2, 4) = u_0(x_1, x_2)$. No boundary conditions apply along the boundary $\partial \Omega$. The mesh and numerical solution are reported in Fig. 8.3 when using the adaptive algorithm with $C = 60$, $TOL = 0.05$, $\tau = 0.001$. Convergence of the adaptive algorithm is checked at final time ($t = 4$) in Fig. 8.4 ($C = 60$) and Fig. 8.5 ($C = 240$), when linear interpolation [7] and a constant time step ($\tau = 0.001$) are used. Computations with conservative interpolation [3] and $C = 60$ is proposed in Fig. 8.6, compare the results with those of Fig. 8.4. Again, the results show that the method used to interpolate the numerical solution between meshes is critical. Finally, the influence of the time step is investigated in Fig. 8.7. It is shown that, for the most precise computation, namely $C = 60$, $TOL = 0.025$, the error due to time discretization is not negligible. Therefore, it seems unavoidable i) to derive a posteriori error estimates for the fully discrete finite element/Crank-Nicolson scheme ii) to use an adaptive algorithm in space and time. This has been done by the second author for the heat equation in [15] and is beyond the scope of the present paper in the framework of the transport equation.
Fig. 8.3. Stretching of a circle in a vortex flow when \( C = 60 \). Mesh and numerical solution at time \( t = 0, 1, 2, 3, 4 \), \( TOL = 0.05 \), \( \tau = 0.001 \) with linear interpolation [7].
Fig. 8.4. Stretching of a circle in a vortex flow when $C = 60$, with linear interpolation [7]. Plot of $u_h(x_1, 0.75, 4)$ with respect to $x_1$, with several values of $TOL$. 

Fig. 8.5. Stretching of a circle in a vortex flow when $C = 240$, with linear interpolation [7]. Plot of $u_h(x_1, 0.75, 4)$ with respect to $x_1$, with several values of $TOL$. 

$TOL = 0.1$, $TOL = 0.05$, $TOL = 0.025$.
Fig. 8.6. Stretching of a circle in a vortex flow when $C = 60$, with conservative interpolation [3]. Plot of $u_h(x_1,0.75,4)$ with respect to $x_1$, with several values of $TOL$.

Fig. 8.7. Stretching of a circle in a vortex flow when $C = 60$ and $TOL = 0.025$, with conservative interpolation [3]. Plot of $u_h(x_1,0.75,4)$ with respect to $x_1$, with several values of $\tau$. 

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Acknowledgements. Frédéric Alauzet is acknowledged for providing the program corresponding to conservative interpolation [3].

REFERENCES


